Revisiting University Students' Knowledge that Involves Basic Differential Equation Questions

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This study documents the extent to which university students utilize diverse representations and mathematical processes to interpret and respond to a set of questions that involves fundamental concepts in the study of differential equations. Results indicate that students' idea to solve a differential equation is reduced to the application of proper solution methods to a certain type of equation differential expressions. Thus, instructional activities should promote the students' use of several representation systems in which they can reflect on the various aspects associated with the concept itself, the solution methods, procedures, and the corresponding meaning and connections among those representations.

Keywords: Differential equations; Meaning; Representations; Solution methods

Conocimiento de los Estudiantes Universitarios con Respecto a Preguntas que Implican Ecuaciones Diferenciales: una Revisión

Este estudio muestra hasta qué punto los estudiantes de universidad utilizan diferentes procesos y representaciones matemáticas para interpretar y responder a un grupo de cuestiones que incluyen conceptos fundamentales relacionados con el estudio de las ecuaciones diferenciales. Los resultados obtenidos indican que la idea que tienen los estudiantes de resolver una ecuación diferencial, se reduce a la aplicación de una serie de métodos. Así, la instrucción debería promover el uso de diferentes sistemas de representación que les permita reflexionar sobre varios aspectos asociados al concepto, los métodos de solución, los procedimientos y los significados y conexiones entre las representaciones utilizadas.

Términos clave: Ecuaciones diferenciales; Métodos de resolución; Representaciones; Significado

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Traditionally the teaching of differential equations has been undertaken with an algorithmic focus in the sense that some guidelines have usually been taught so that these equations are then classified as certain types and solved, sometimes using certain techniques that lead to the solution either explicitly or implicitly. However, what do students deem to be relevant when using these techniques or algorithms to solve problems? Will the students remember them and will they be able to use them when needed? Do they recognize that it is not possible to give an explicit or implicit expression for the solutions for most differential equations? How do they behave when faced with a differential equation which cannot be solved by using the methods they have studied?

In this study we document and analyze students' types of behavior when attempting to solve some differential equation problems which are presented from a different perspective than they normally appear during the process of instruction. We describe the processes associated when solving these tasks, as well as the strategies for solving problems that students use when carrying out the activities set.

Our research responds to four main questions:

- Do students use knowledge gathered during their previous studies (meaning of the derivative, function concept, graphics representations, etc.) to answer questions on differential equations that do not necessarily require methods belonging to this field?
- What use do they make of the various systems of representation?
- What influence does the wording of the question have on students' mode of approaching it?
- What types of strategies and representations do students use when faced with contextualized problems?

CONCEPTUAL FRAMEWORK

The learning or development of mathematical knowledge is a process that demands continual reflection on the part of students to help them represent and examine mathematical concepts from different points of view and lead them to construct a network of relations and meanings associated with this concept (Camacho, Depool, & Santos-Trigo, in press). Development of this process of construction depends directly on the systems of representation used and the coordination between them (Duval, 1993). On the other hand, the learning of a mathematical concept is directly related to the activities undertaken to solve problems (Santos-Trigo, 2007). Problem solving, then, should form a major part of teaching. In this context, the student formulates questions, puts forward conjectures, seeks different ways of validating them, and communicates his or her answers or results in a suitable language. Thurston (1994) stated that the comprehension of the concept of derivative involves thinking of diverse ways to define, operate, represent, and interpret its meaning:

> *Infinitesimal: The ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.*
>
> *Symbolic: The derivative of* x^n *is* nx^{n-1}*, the derivative of* $Sin(x)$ *is* $Cos(x)$*, the derivative of* $f \circ g$ *is* $f' \circ gg'$*, etc.*
>
> *Logical:* $f'(x) = d$ *if and only if for every* ε *there is a* δ *such that when* $0 < |\Delta x| < \delta$, $\left|\frac{f(x+\Delta x) - f(x)}{\Delta x} - d\right| < \varepsilon$
>
> *Geometric: The derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.*
>
> *Rate: The instantaneous speed of* $f(t)$ *when* t *is time.*
>
> *Approximation: The derivative of a function is the best linear approximation to the function near a point.*
>
> *Microscopic: The derivative of a function is the limit of what you get by looking at it under a microscope of higher and higher power.* (p. 3)

Thus comprehending the concept of derivative or those that involve the study of differential equations requires or demands that students relate and transit, in terms of meaning, through the ideas and representations associated with each way of thinking about those concepts.

Based on these premises, we can see that the understanding of a mathematical concept passes through various stages or phases, among which there is: (a) The phase where the student understands the definition of the concept itself, (b) the phase where this concept is used algorithmically, and (c) the phase where the concept is recognized as an instrument to solve problems. Along the route taken for constructing mathematical knowledge, it is important to identify the previous knowledge and forms of thinking that students use when attempting to understand mathematical ideas and solve problems.

The concept of solving a differential equation and the direction field associated with it are some of the meanings that are closely connected with the concept of differential equation. The graphic nature of the direction field —geometric meaning in the sense attributed by Thurston— and the traditionally algebraic focus from which differential equations are taught —symbolic meaning in the sense attributed by Thurston— suggest that we need to analyze the balance or complementariness of the relations between the various systems of representation. The understanding of the concept of the solution of an ordinary differential equation (ODE) develops as the definition of the concept joins up with other elements, among which we can find those that can be seen in Figure 1. Also, the concept of direction field associated with an ODE includes two related phases that are different from the cognitive point of view: Interpretation and representation.

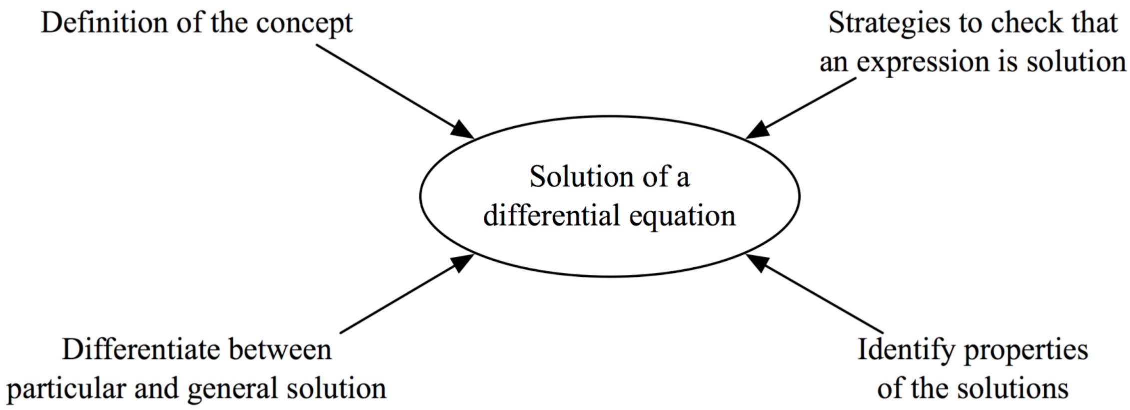

Figure 1. Solution of a differential equation

METHODOLOGY

A total of 21 students took part in this study, 10 of whom were studying for a mathematics degree and 11 were studying the physics degree (in the University of La Laguna). The main difference between the two groups was the instruction they received: The mathematics students received a more theoretical approach while the physics students a more practical instruction. This difference is due in part to the nature of the subjects that the students were taking. The mathematics students were studying a fifth semester subject devoted solely to differential equations, while physics students covered the material in their second semester and the subject that they were taking covered differential equations as well as other calculus concepts.

The analysis of the strategies used by students to solve the activities set was made based on a questionnaire designed specifically for this purpose. The questionnaire is made up of 11 problems which can be solved using several methods or for which something more than the application of rules, formulas or algorithms is required (Santos-Trigo, 2007). The selection of these problems was made taking into account the results given in the literature review and some of them were chosen from textbooks used in differential equations courses. Other tasks were specifically designed in order to respond to the main questions of our study. The questionnaire includes activities where students needed to use properly algebraic and graphic systems of representation, both separately and together, and it was necessary for students to use their knowledge of solving of problems set in a real context. Problems were classified into four types.

Problems of Type 1

These questions require knowledge of the concept of solution. This type of question is used to check whether an algebraic expression is a particular or general solution to a differential equation (Q3, Q4 and Q11) and to analyze some general properties of the solutions in function of the terms of this expression (Q5). We can observe Q3 and Q5 in the following lines.

Q3 *Say whether the following statements are true or false and give reasons for your answer:*

a) The function $y = e^{\int e^{t^2} dt}$ *is a solution for the differential equation* $\frac{dy}{dt} = 4e^{t^2} y$.

b) The function $y = f(x)$ *which allows* $-x^3 + 3y - y^3 = C$ *is a solution for the differential equation* $\frac{dy}{dx} = \frac{x^2}{1-y^2}$.

Q5 *Say whether the following statement is true or false and give reasons for your answer: "Take the first order differential equation* $y'(x) = f(x,y)$. *If the function* $f(x,y)$ *is defined as* R^2, *solutions for the differential equation will also be defined as* R^2*".*

Problems of Type 2

Solutions of this type of question can be achieved through the use of logical reasoning (Q1) or using simple algebraic methods (Q2). This type of question implies graphic representation of elemental functions, but do not involve either the construction or interpretation of the direction field or the interpretation of data from or towards a mathematical context. An example of this is Q1.

Q1 *Represent graphically some solutions to the following equations:*

a) $\frac{dy}{dx} = 0$; $x \in [0,2]$

b) $\frac{dy}{dx} = \cos x$

Problems of Type 3

Questions where solving requires representation and/or interpretation of the direction field of a differential equation (Q6, Q8 and Q10). We can see the statement of Q10 in the following lines.

Q10 *Draw the direction field for the differential equations* $\frac{dy}{dx}=1$ *and, based on this, solve the following initial value problem* $\begin{cases} \frac{dy}{dx}=1 \\ y(-2)=4 \end{cases}$

Problems of Type 4

Activities where it is necessary to interpret information supplied in algebraic or graphic terms, in a real context, or vice versa (Q7 and Q9) are identified as problems of type 4. We can observe Q7 as an example.

Q7 *We know that the population of a city grows constantly over time, substantiating the differential equation* $\frac{dP}{dt}=K,\ K>0$. *If the population has doubled in 3 years, and in 5 years it has reached a total of 40,000 inhabitants, how many people lived in the city at the beginning of the five-year period?*

DISCUSSION AND RESULTS

We mainly focus here on three of the questions taken from the questionnaire in order to analyze the processes of solution followed by the students when the tasks are framed in different types of contexts. To this end, we analyze the answers from students for Questions Q1*a* (type 2), Q7 (type 4) and Q10 (type 3) from the questionnaire. We represent the mathematics students as MSi (i = 1,…, 10) and the physics students as PSj (j = 1,…, 12). We eliminated student PS10 from our analysis because this student did not manage to answer any of the questions in the questionnaire.

Those students who solved tasks Q1a and Q7 but not Q10 (MS3, PS7, PS10, and PS12) share the common characteristic of having shown that they know some algebraic methods for solving differential equations but that they failed in representing any of the direction fields asked for in the questionnaire and they also failed in making mathematical interpretations. Another characteristic that can be underlined regarding these four students is that, while they indeed attempted to solve both questions Q1a and question Q7, they did so without using the same form of reasoning. While they all solved the equation of the problem Q7, taking it as one of separate variables, only MS3 used this solution strategy for Q1a. Moreover, of these four students, only PS12 correctly solved the differential equation. The other three students omitted the integration constant when applying the method of separate variables, an error that they did not make when they were solving the equations in problems Q1 and Q2.

Student PS4, who was the only one who dealt with the question Q10 and not Q7, answered hardly any of the questions set in the questionnaire; she only answered the problems Q1 and Q10 (see Figures 2 and 3). In spite of this, she was able to find a particular solution to each of the differential equations set. So, in the question Q10, although she was asked to solve the problem based on the direction field, this student expressed the solution for the initial value problem using logical reasoning in order to find that it would be a linear function, and guesswork in order to find the constant that was missing. This is an example of how intuitive the ordinary differential equation is.

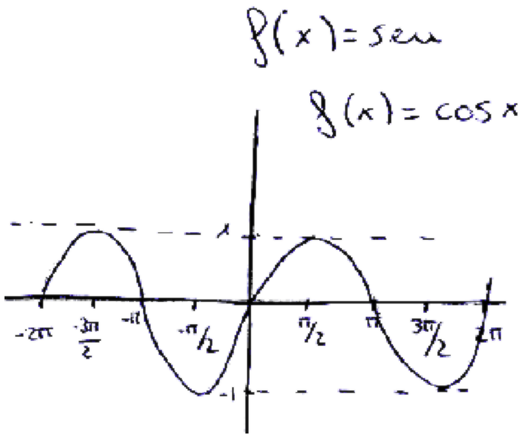

Figure 2. Answer from PS4 to Q1

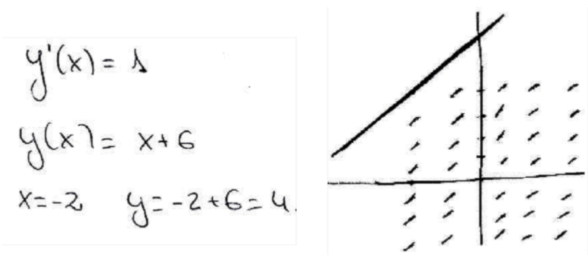

Figure 3. Answer from PS4 to Q10

We now analyze the answers from those students who attempted to answer the three questions that we are focusing on. Given the characteristics of the questions' text, Q10 induces use of the direction field associated with a differential equation in order to find a solution, which clearly distinguishes it from Q1a and Q7. We might think, then, that this problem is going to be solved by students in a way different from that used in the other two questions, due precisely to the use of the graphic representation system. However, we find that of the 10 students under study only 2, MS4 and PS9, set about this task using the direction field. The other students' responses depended on an algebraic solution of the equation.

Regarding the strategies used by students to solve the problems Q1a, Q7 and Q10, we find the following types of behavior. Students PS2 and PS11 demonstrated through the questionnaire that they knew some methods for solving differential equations. However, they did not take those methods into account when solving task Q7. PS11 used a rule of three to solve it, while PS2 did not attempt to solve the equation on finding that he cannot translate the data in the problem into mathematical language (Figure 4[1]). This is the only differential equation that this student does not solve, which shows us that the student fails to relate the equation that appears in the text of Q7 with those that the student has solved in the rest of the activities.

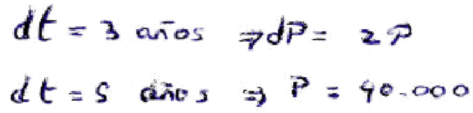

Figure 4. Answer from PS2 to Q7

On the other hand, student MS1 and PS6 found the solutions to the equations in the three problems by either using knowledge they had acquired prior to their study of differential equations or directly, without being able to appreciate the use of specific methods for solving differential equations (see Figure 5[2]).

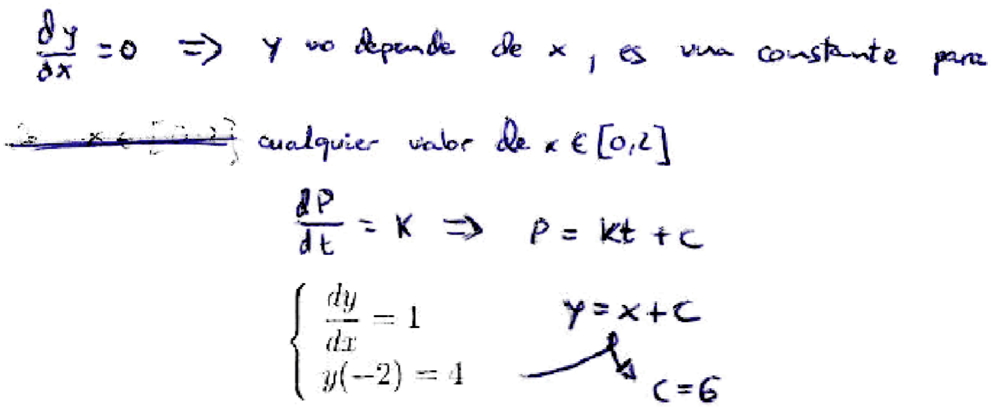

Figure 5. Solution from PS6 to various differential equations

Four students (MS7, MS10, MS9 and PS3), once they had solved the equation $y' = 0$ using concepts and procedures that they had learned prior to their study of differential equations or by expressing the solution of the equation algebraically, without explicitly showing the use of any specific method of solution, then they used the method for solving separate variable equations when they solved tasks Q7 and Q10.

The context of the question did not influence them when they made certain mistakes. In all their answers they avoid the integration constant (MS9 and PS3). However, MS10 only made this mistake in the contextualized problem Q7, while student MS7 found herself in difficulties by not being able to interpret the information on population supplied in the problem text in mathematical terms, which means she was unable to solve Q7 correctly.

[1] "años" means "years" (the editor).

[2] The text says "y does not depend on x, it is a constant for any value of x…" (the editor).

Finally, MS4 and PS9 used different strategies when solving these three activities, limiting themselves to the stipulations of the question texts. PS9, meanwhile, made some modifications when solving the equations in Q1a and Q7, using definite integrals for the latter but using indefinite integrals for the rest of the differential equations solved (see Figures 6[3] and 7).

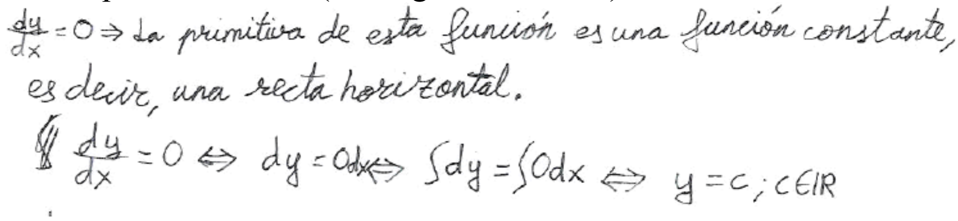

Figure 6. Answer from PS9 to Q1a

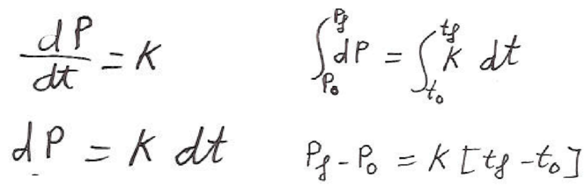

Figure 7. Part of answer from PS9 to Q7

CLOSING REMARKS

Students' answers to the various questions set show once more that they prefer to use the algebraic rather than the graphic and verbal register. This might be a result of the instruction they had received, in which algebraic aspects were predominant, graphic studies were only superficially covered and there was no incentive to find a possible verbal solution (González-Martín & Camacho, 2004). Moreover, the students' deficiencies when undertaking activities associated to problem solving, such as analysis of the problem, decision making and the evaluation of the solution (Santos-Trigo, 2007).

Many of the students conceived the concept of differential equation as an isolated mathematical entity unconnected to other notions they know. For the students, solving a differential equation is merely a matter of finding an implicit or explicit algebraic expression of the solution. So they considered that the relevant information supplied in a differential equation was the information that could lead them to apply some method in order to reach the solution.

It has been also shown that in general, after a certain amount of time in which the student could not remember the methods and that they did not have an understanding of the concept of solution that allows them to solve problems (for example, Q7) without using the algorithmic methods studied.

[3] The text says: "The primitive of this function is the constant function; that is, a horizontal straight line" (the editor).

We consider that introducing concepts based on others already known can allow the student to make connections between the different themes or questions studied. Also, this will permit a broader vision of the concept of differential equation, and would not limit this to the use of certain "tricks" which are easily forgettable and unfruitful. Accepting the system of graphic representation as legitimate in the process of solution can broaden the understanding of the concept. Teaching based on problem solving give the students the opportunity to observe the need to take into account and work with different registers of representation. This will motivate them to tackle questions related to mathematics in general and, in particular to differential equations in a more open-minded and complete form, greatly increasing their chances of success when it comes to solving problems.

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