DEVELOPING A FRAMEWORK FOR EVALUATING STUDENT'S UNDERSTANDING OF FIGURAL PATTERN GENERALIZATION

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Figural patterns have a unique capacity to promote functional thinking. This study aimed to identify the mental constructs of 7th-grade students in Figural Pattern Generalization (FPG) by using the Action, Process, Object, Schema (APOS) theory in order to develop a framework for evaluating students' understanding of FPG. A sample of 220 students completed a test designed based on the APOS framework and 19 students participated in a semi-structured interview. Results showed that there are emergent and partial action levels before the action stage and pre-emergent, and partial process/object levels before the process/object stage.

Keywords: APOS theory; Developing framework; Figural Pattern Generalization; Representation of structures

Desarrollo de un marco para evaluar la comprensión del estudiante en la generalización de patrones figurativos

Los patrones figurativos tienen una capacidad única para promover el pensamiento funcional. Este estudio tuvo como objetivo identificar las construcciones mentales de los estudiantes de 7º grado en Generalización de Patrones Figurativos (FPG) mediante el uso de la teoría de Acción, Proceso, Objeto y Esquema (APOS) para desarrollar un marco para evaluar la comprensión de la FPG. Una muestra de 220 estudiantes completó una prueba diseñada en el marco APOS y 19 estudiantes participaron en una entrevista semiestructurada. Los resultados mostraron que existen niveles de acción emergentes y parciales antes de la etapa de acción y niveles de proceso/objeto pre, emergentes y parciales antes de la etapa de proceso/objeto.

Términos clave: Generalización de patrones figurativos; Marco de desarrollo; Representación de estructuras; Teoría APOS

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Desenvolvimento de uma estrutura para avaliar a compreensão dos alunos sobre a generalização de padrões figurativos

Os padrões figurativos têm uma capacidade única de promover o pensamento funcional. Este estudo teve como objetivo identificar as construções mentais dos alunos do 7.º ano na Generalização de Padrões Figurativos (FPG), utilizando a teoria das Acçõ, Process, Object e Esquema (APOS) para desenvolver um quadro de avaliação da compreensão da FPG. Uma amostra de 220 alunos completou um teste concebido no âmbito da APOS e 19 alunos participaram numa entrevista semi-estruturada. Os resultados mostraram que existem níveis de ação emergentes e parciais antes da fase de ação e níveis de processo/objeto prévios, emergentes e parciais antes da fase de processo/objeto

Palavras-chave: Generalização de padrões figurativos; Quadro de desenvolvimento; Representação de estruturas; Teoria APOS

Mathematics is the knowledge and language of patterns (Copley, 2000). Thinking about patterns helps students to create mathematical knowledge. According to National Council of Teachers of Mathematics (NCTM, 2000), pattern generalization prepares the primary and secondary students for functional situations, familiarity with variable concepts, and moving toward algebraic thinking. In Figural Pattern Generalization (FPG) tasks, it is necessary to mark the constant and changing components to independent and dependent variables, so a direct formula (symbolic or non-symbolic) can be derived from the given stages as a functional relationship. In fact, patterns are a concrete representation of functional relationships. Therefore, FPG have a unique capacity to promote functional thinking (Markworth, 2010; Rivera, 2013) as a gateway into algebraic thinking (Carraher & Schliemann, 2019). Thus, studying and exploring the mental constructs of students in FPG may help to facilitate this promotion. In this research, we use APOS —Action, Process, Object, Schema— theory (Dubinsky, 1991) as a theoretical approach to identify the mental constructs of 7th grade students in FPG. Additionally, we use Mulligan and Mitchelmore (2009) levels for representation of structure to classify the identified levels of APOS theory in FPG. This research was conducted to justify the existence of levels between stages within APOS theory and name these levels to develop a framework for accurate evaluating students' understanding of FPG.

BACKGROUND

A bulk of research has been carried out in the area of generalization of numerical and figural patterns. Some of these studies have addressed the generalization strategies of students (Chua & Hoyles, 2014; Samson, 2011). Much researches focus on effective factors in generalization, including pattern task characteristics and designing appropriate questions (Chua, 2009; Chua & Hoyles, 2012; Lannin et al., 2006; Samson, 2007). Moreover, other studies have focused on the pattern generalization related to the development of algebraic thinking and functional thinking (Blanton & Kaput, 2011; Kaput et al., 2007; Markworth, 2010; Oliveira et al., 2021; Smith, 2008; Wilkie & Clarke, 2016). There are also a few studies that examined the generalization of patterns with the APOS framework and adapted the Action- Process-Object-Schema stages to the pattern generalization (Sutarto et al, 2016; Sutarto et al., 2018; Yuniati et al., 2020). Rivera (2013) synthesized at least 20 years of research studies on pattern generalization and organized a framework that took into account various aspects of pattern generalization. One aspect that Rivera has mentioned related to the types of structures. Mulligan and Mitchelmore (2009) extrapolated progressions in structure that they inferred on children who deal with patterning tasks. Our research used the representation of structure of Mulligan and Mitchelmore (2009) to classify identified levels of mental constructs of 7th grade students in FPG by using APOS theory. The reason for conducting this study is due to the lack of research in this field that there is no attention to identify the levels between stages in concept construction of FPG. Therefore, this study shows the power of APOS theory for identifying mental constructs in the FPG concept and designing a more accurate framework for evaluating students' understanding of FPG.

APOS Theory

Action-Process-Object-Schema (APOS) Theory explains how individuals make meanings of mathematical concepts by constructing and using certain structures (Dubinsky, 1991). These structures are created through the mechanisms of interiorization, coordination, reversal, encapsulation, de-encapsulation, thematization, and generalization (Arnon et al., 2014). In the action stage, a concept is perceived as an externally driven action outside the mind, which is performed according to the rigid application of explicit or memorized step-by-step instructions. The steps cannot be guessed and imagined, and none can be skipped. An action is perceived as a mechanical procedure and lack meaningful internal relations to other mathematical ideas.

Once the individuals repeat and reflect on the actions instead of relying on external signs, they focus on their internal control and are transmitted to the stage of process. The features of this stage are the ability to imagine the steps without explicitly performing them, the ability to skip the steps and reverse them. Interiorization is the mechanism that provides this mental transmission (from action to process), which enables the person to act consciously, reflect upon, and combine it with other actions. Processes may be reversed and coordinated with other processes. As the range of applicability of a process increases, individuals may feel the need to apply actions on the process to cope with new situations. In this case they may become aware of the process as a totality and realize that transformations can act on the totality.

When individuals apply or can imagine applying such transformations, then it is said that the process has been encapsulated into a cognitive object (Arnon et al., 2014). When a process is encapsulated into an object, it can be returned to the process in which the object emerged by the mechanism of de-encapsulation. The interaction of the elements mentioned above gives rise to schema. A mathematical schema is considered as a coherent collection of actions, processes, and objects, and other previously constructed schema (Baker et al., 2000) that contain the description, organization, and exemplification of the mental structures that an individual has constructed regarding mathematical concept. Coherence of schema is indicated by the individual's ability to determine what is in the scope of the schema and what not is (Arnon et al., 2014).

Levels in APOS theory

In a number of APOS theory-based studies (Arnon, 1998; Dubinsky et al., 2013; Weller et al, 2009) there are suggested levels between stages of Piaget (1975), and Dubinsky et al. (2013) have described three characteristics in relation to levels and stages: (1) Individuals cannot skip from a stage; if this happens, they will lose coherent understanding of the concept. Therefore, the stages are sequential and each stage is necessary for the next stage. (2) A level may or may not exist in the data derived from an individual; individuals may move to the next level or stage quickly and the levels may be not visible. (3) The stages are defined in terms of the basic structures that are general and do not depend on a specific mathematical content, but the levels depend on the particular subject and data from different individuals will be different for different concepts (Arnon et al., 2014).

Figural Patterns Generalization and Representation of Structure

Generally, patterns occur as two types: numerical or figural. A numerical pattern is a series of numbers verifying certain a rule between all numbers. Figural pattern can be defined as “a sequence of figures in which the objects in the figure change from one term to the next, usually in a predictable way” (Huntzinger, 2008, p. 280). These objects have common properties that are not inherently a priori but make sense within some interpreted structure, depending on an individual learner’s knowledge and experiences (Rivera, 2011). Patterns generalization means that learners identify commonalities in a particular case or doing abstraction in the sense of “seeing a generality through the particular” (Mason, 1996, p. 65) and then they develop these commonalities to the next cases.

Figural patterns are grouped into two main categories of repetitive patterns and incremental/decreasing patterns. In incremental/decreasing patterns, objects in the form of a case change to another in a systematically and predictable manner. Incremental figural patterns can appear in different structures. Their structures can be expressed in the form of a linear function, quadratic or two-variable function, and so on. In fact, the structure of a mathematical pattern is a way that it is organized and is expressed in the form of generalization (Mulligan & Mitchelmore, 2009). Many empirical studies with several different cohorts of Australian children illustrate how young learners' structural representations transition from the pre structural stage to the emergent stage and then partial stage, and then finally to the full stage of structural development (Mulligan & Mitchelmore, 2009; Mulligan et al., 2004; Papic et al., 2009, 2011). These four broad stages of structural development are as follows:

- Pre structural stage: Students in this stage tend to produce idiosyncratic features in their representations that have no evidence of numerical or spatial structure.
- Emergent stage: Representations of student show some relevant elements of the given structure that are oftentimes influenced by what they find meaningful and relevant, but their numerical or spatial structure is not represented.
- Partial structural stage: Representations show most relevant aspects of numerical or spatial structure, but the representation is incomplete.
- Stage of structural development: Representations correctly integrate numerical and spatial structural features.

Mason et al. (2009) stated that mathematical structures mean the identification of general properties which are instantiated in particular situations as relationships between elements or subsets of elements of a set. They also state that structural thinking is not a mere act of recognizing relationships and properties as it is more about employing it in their own thinking relative to mathematical objects (e.g., figural and numerical patterns, functions, sets). The mathematical structure is a prerequisite for structural thinking, which can be linked to cognitive structures that produce schemas that are essential in math thinking and successful learning (Gronow, 2015). Therefore, structural thinking in pattern tasks is not just about identifying relationships and writing them, but these relationships must be used to distinguish structure of patterns and properties of change in patterns, and help to identify what kind of pattern belongs to which category and why. Jones and Bush (1996) state that structural thinking help students to understand and answer "why" questions (specifically at the schema stage in APOS theory).

RESEARCH QUESTION

In this paper, we answered to the following questions: What are mental constructs of students in FPG according to APOS theory? More specifically, the questions addressed are the following: What are the levels before the Action stage? What are the levels before the Process stage? What are the levels before the Object stage?

RESEARCH METHOD

The present research is part of a broader research project which uses quantitative and qualitative (mixed) design. The research framework utilizes APOS theory and was conducted in three steps. In the first step, mental constructions of FPG were analyzed using the background of FPG, analysis of the FPG concept itself by researchers, and the researchers' experiences, based on APOS theory, and the analytical model of FPG was designed that describes the mental constructions that students need to learn FPG concept. In the second step, a test about FPG that included 4 tasks (24 items) was designed based on analytical model to gathering quantitative data about mental constructs of students. The validity of the test was confirmed by three experts in mathematics education and four experienced teachers. Internal consistency of questions was estimated with Cronbach's Alpha and reported to be 0.69. In this research, 220 7th grade students were selected based on the Cochran formula for determination of sample size and answered to the test in 30 minutes. 32 answer sheets were discarded because they were white or illegible, and 188 answer sheets were analyzed based on analytical model of FPG designed in first step. At the third step, a sample of 19 students from the same population took part in the interview sessions. In this step, Mulligan & Mitchelmore's (2009) levels for representation of structure were used to classify students' responses in different levels before stages. Based on the school's scores, students were identified as one of three groups of proficiency: lower average, middle average, and above average.

Instruments

In this study, an instrument designed that contained 6 tasks broken down into 30 items; these items are categorized in Table 1. Tasks 1 - 4 (items 1-24) were used in the test at second step of research. Task 5 (items 25-28) and task 6 (items 29-30) were used in the semi-structured interview with 19 students (see Appendix 1) at the third step. In order to design the instrument, the researchers studied the affective factors and characteristics in generalization and pattern tasks (Chua, 2009; Chua & Hoyles, 2012; Samson, 2007). Researchers focused on two features of FPG tasks: Researchers focused on the number of variables and whether the function was linear or nonlinear (Chua, 2009); task1 is one variable-linear, task2 is one variable- nonlinear, task3 is two variable- linear and task 4 is two-variable, nonlinear. Based on stages of APOS, items of FPG were designed (See Table 1).

Table 1
Describing the instruments for identifying APOS stages of students

Number of items	Content	Expectations
1-7-13-19	Drawing next pattern (close generalization).	Drawing correctly the figure with all details.

Table 1
Describing the instruments for identifying APOS stages of students

Number of items	Content	Expectations
2-8-14-20	Explaining the next pattern without drawing.	Explaining the figure structurally.
3-9-15-21	Reversing the steps to get the figure number.	Reversing the steps and answer the reverse item.
4-10-16-22	Far generalization.	Answering the far generalization item that requires jumping from the steps.
5-11-17-23	Writing an algebraic relation.	Writing the relation completely and manipulating it to answer the reverse question.
6-12-18-24	Manipulating with relation.	Designing a structural figural pattern for the given relations.
25-26-27-28	De-encapsulating relation to their process.	Understanding the equivalence of two relations and showing the equality between them.
29-30	Determining type and property of figural pattern, reasonably.	Distinguishing between one- and two-variable figural patterns reasonably. Distinguishing type of functional relations of figural patterns reasonably.

Data Analysis

The items of tasks in the test designed based on features of the stages of APOS theory and answer sheets of students coded by these features. If they answer to any item correctly code1 were allocated and else of code 0. For example, in the items of 1,7,13 and 19 if students draw correctly the figure with all details code 1 were allocated them.

- The features of action stage of APOS are as follows: a concept is perceived as an externally driven action outside the mind (in FPG, Drawing the next pattern).
- The features of process stage of APOS are as follows: the ability to imagine the steps without explicitly performing them (in FPG, Explaining the next pattern without drawing), the ability to skip steps (in FPG, Answering the

far generalization item that requires jumping from the steps), and reverse steps (in FPG, Reversing the steps to get the figure number).

- The features of object stage of APOS are as follows: become aware of the Process as a totality (in FPG, writing algebraic relation), realize that transformations can act on the totality (in FPG, manipulating with relation of patterns), and object can be returned to the process by the mechanism of de-encapsulation (in FPG, de- encapsulating the relation to their process).

Mulligan & Mitchelmore's (2009) levels for representation of structure were used to classify students' responses in different levels before stages. For example, according to the *Emergent Stage*, students' representations show some relevant elements of the given structure that are oftentimes influenced by what they find meaningful and relevant, but their numerical or spatial structure is not represented. If the student draws a figure with any structure at the items 1,7,13 and 19, we name this level as *Emergent Action* level (EA), before action stage in APOS theory. Table 2 shows the levels that we expect to occur between stages in concept construction of FPG.

Table 2
Levels and stages in APOS theory based on Mulligan & Mitchelmore's (2009) levels for representation of structure

Levels/ stages	Indicators
PRA	There are no signs associated with the given figural pattern.
EA	There are some signs associated with a given figural pattern, but no structure of the figure is seen.
PA	There are many signs associated with the structure of a given figural pattern but not all of them.
A	The figural pattern is drawn with all the details and correctly.
PRP	There is no sign of a process. The responses of the process part are not related to the question.
EP	Students write numerical relations for figural pattern with conjecture or try and error, or describe how the figure has been constructed, without any structure.
PP	Answers of students have most of the signs of process stage, but not complete. They describe the structure of the pattern, but still not able to reverse the figure structure or far generalization.
P	There are all signs of the process in responses of student. They explain the structure of the figure, reverse the steps and respond to the inverse question, and answer the far generalization question that requires the jump from the steps, correctly.

Table 2
Levels and stages in APOS theory based on Mulligan & Mitchelmore's (2009) levels for representation of structure

Levels/ stages	Indicators
PRO	There are still no signs of reaching to the object stage. Students write the relation verbally and can't use the symbol.
EO	The student writes the relation with verbal-symbolic integration.
PO	Students write the relationship and makes meaningless manipulations. In reversing the relationship to the figural pattern, they only work with the numbers derived from the relation; Draw the figure according to the number obtained from the relation and without regard to the structure that exist in relation.
O	Students fully write the relation and manipulate it to find the answer to the reverse question. They design a structural figural pattern for a given relation.
S	Students can distinguish among different figural patterns with different structures and they can recognize the type of corresponding functional relationship and express their reason.

FINDINGS

The results derived from the review of responses of sample to the test at the second step of research are presented in Table 3. It shows the difference among the number of students who reached the Action, Process, and Object stage.

Table 3
Percentages of correct responses of sample in the test at the 2nd step of research

	Action stage		Process stage		Object stage	
Task Number	Drawing next pattern	Explaining the next pattern	Reversing the steps	Far generalizing	Writing algebraic relation	Manipulating with relation
Task 1	88.8%	63.8%	66.7%	59.6%	52.7%	5.6%
Task 2	72.9%	30.3%	17.5%	21.3%	18.1%	0%
Task 3	91.5%	44.1%	40%	56.9%	44.7%	2.7%
Task 4	76.1%	20.2%	20%	22.9%	12.8%	1.8%
Average percent	82.3%	39.6%	36.1%	40.2%	32.1%	2.5%

What are mental constructs of students in FPG according to APOS theory?

What are the levels before the Action stage?

Table 3 shows that about 18% of students did not reach the action stage. Studying the responses of students who did not reach this stage showed that they are at different levels as it is explained below.

In the *Pre Action* level (PRA), there are no signs associated with the given figural pattern. We expected to see this level in the students' answers, which we did not see in our statistical sample. In turn, in the *Emergent Action* level (EA), there are some signs associated with a given figural pattern, but no structure of the figure is seen. Figure 1 shows that the student draws figure with any structure (see Task 1 in Appendix 1).

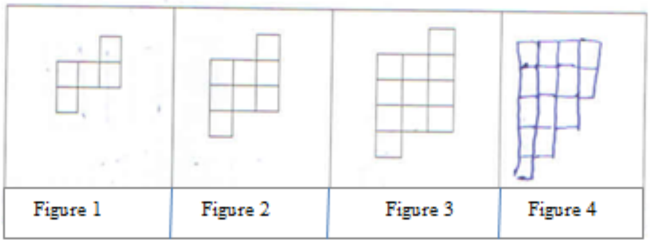


Figure 1. Example of a student's answer in level of Emergent Action

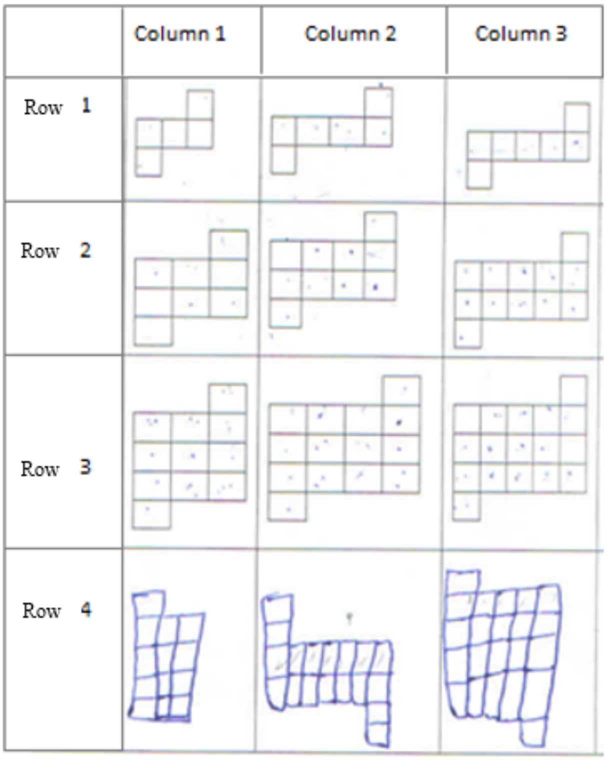


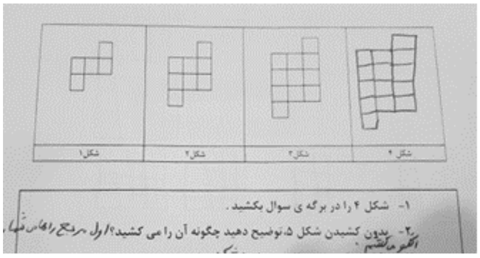
Figure 2. Example of a student's answer in level of Partial Action

In the *Partial Action* level (PA), there are many signs associated with the structure of a given figural pattern but not all of them. Figure 2 shows this level of student (see Task 4 in Appendix 1). Finally, in the *Structural Development* level or *Action stage* (A), the figural pattern is drawn with all details and correctly. The correct answers to tasks 1, 2, 3, 4 at items 1, 7, 13, 19 are examples at Structural development level or Action stage.

What are the levels before the Process stage?

As Table 3 shows, students who succeeded in drawing the next pattern (82.3%) were not necessarily successful in explaining the structure of the pattern (39.6%), reversing the process (36.1%) or in far generalization (40.2%).

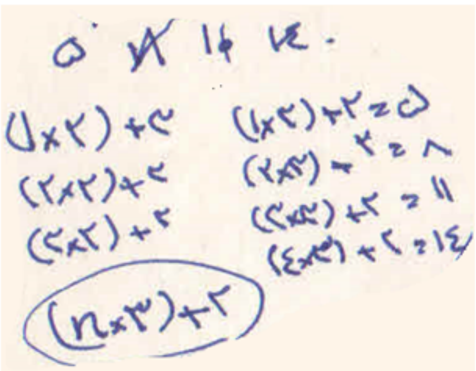
In the *Pre Process* level (PRP), there is no sign of a process. The responses of the process part are not related to the question. Figure 3 shows that the student has answered the question without any signs to reach the process stage (see the item 2 of Task 1 in Appendix 1). On the other hand, in the *Emergent Process* level (EP), the student writes numerical relations for the figural pattern with conjecture or trial and error, or describes how the figure has been constructed without any structure. Figure 4 shows this level (see Task 1 in Appendix 1).



2- Without drawing figure 5, describe how you would draw it?

Answer: at first count squares and then draw the pattern respectively.

Figure 3. Example of a student's answer in level of Pre Process



5 8 11 14

(1×2)+3 (1×3)+2=5
(2×2)+3 (2×3)+2=8
(3×2)+3 (3×3)+2=11
(4×3)+2=14

(n×3)+2

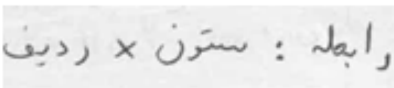
Figure 4. Example of a student's answer in the level of Emergent Process

In the *Partial Process* level (PP), the answer of the student has most of the signs of process stage, but not completely. Students describe the structure of the pattern, but still are not able to reverse the figure structure or far generalization. To conclude, in the *Structural development* level or *Process stage* (P), all signs of the process are present in the response of student. He/she explains the structure of the figure, reverses the steps, and responds to the inverse question, and correctly answers the far generalization question that requires the jump from the steps. The correct answers to items of process stage (see Table 1 above) are examples for this level.

What are the levels before the Object stage?

According to the data from Table 3, there is a large difference between the percentage of students who were successful in writing the symbolic relationship (32.1%) and the students who were successful in manipulating with a written relationship (2.5%). The ability to manipulate an object is a feature of the object stage, which is still not accessible to a large number of students but there are signs of the appearance of the object stage. Student responses have shown the levels that according to Mulligan and Mitchelmore (2009) are as follows:

In the *Pre Object* level (PRO). There are still no signs of reaching to the object stage. The student writes the relation verbally, but she cannot use the symbol. An example is shown in Figure 5 (see the item 17 of Task 3 in Appendix 1). In turn, in the *Emergent Object* level (EO), the student writes the relation with verbal-symbolic integration. Figure 6 shows this level (see the item 11 of Task 2 in Appendix 1).



row× column : Relation

Figure 5. Example of a student's answer at the level of Pre Object (The right picture is translated from left picture)

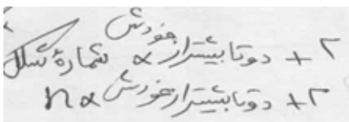


Figure number × two more than figure number+2
n × two more than n+2

Figure 6. Example of a student's answer at the level of Emergent Object (The right picture is translated from left picture)

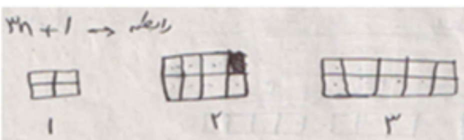


Figure 7. Example of a student's answer at the level of Partial Object (The right picture is translated from left picture)

In the *Partial Object* level (PO), the student writes the relationship and makes meaningless manipulations. In reversing the relationship to the figural pattern, he/

she only works with the numbers derived from the relation; Draws the figure according to the number obtained from the relation and without regard to the structure that exists in the relation Figure 7 shows this level (see Task 25 in Appendix 1). To conclude, in the *Structural Development* level or *Object* s*tage* (O), the student writes the relation correctly and manipulates it to find the answer to the reverse question. The student designs a structural figural pattern for a given relation.

In the second stage of the study, the levels and stages of FPG concept are studied at a closer look using a semi-structured interview with 19 students. In this section, interviews with two students are discussed to confirm the existence of different levels of the FPG concept.

Interviewer: When you encounter such tasks (refers to task 1), what do you do to answer these questions? (Students 1 and 2 answer this question)

Student S1: First, I look carefully at the figures. Then I think a few moments about the next figure and after that I draw the figure. After drawing the next figure, I convert the figural pattern into a numerical pattern and look for the relation between them. Of course, sometimes the relation is hidden in the figures, and with enough concentration, we can see the relation of pattern accurately.

S1 is a student who has reached the object stage and when she is asked to design a figural pattern for the given relations (see Task 25 in Appendix 1), she structurally draws figures and communicates meaningfully between the components of the relation and the figures (see Figure 8). S1 is unable to answer the question like "what is the difference between the quadratic figural pattern and the linear figural pattern?" which relates to the Schema Stage.

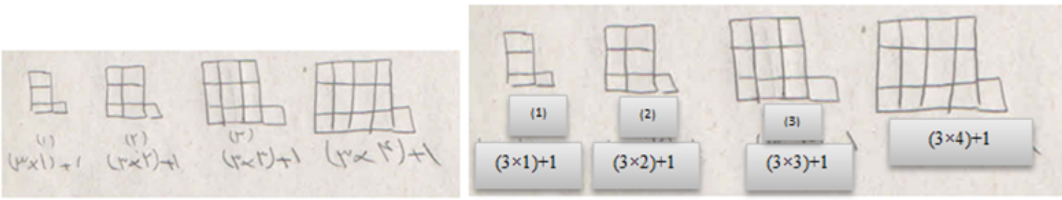


Figure 8. Examples of S1's answers to pattern tasks; Task 25 (The right picture is translated from left picture)

Student S2: We first count the existing shapes and second put the number together and write the number of add-ons, and then write n multiply by x. I try n multiples from the first (1, 2, etc.) to get one right.

S2 is a student who has not reached the Process Stage in pattern tasks. She does not pay attention to the structure of figures and by conjecture breaks down the extracted numbers from the pattern to write a relation that applies to all patterns. In a linear task, although the relation is correct, her understanding of the relation has not reached the process stage. Figure 4 above visualizes the result (see Task 1 in Appendix 1).

CONCLUSION

FPG has a unique capacity to enhance functional thinking (Markworth, 2010). Rivera (2013), after studying 11-year-old students longitudinally, states that students were able to solve modeling/function questions after working with pattern generalization tasks without being trained in modeling and functions. So, without any doubt, attention to figural pattern generalization will promote functional thinking. This research was carried out to identify mental constructions of students regarding FPG concepts for developing a framework to accurately evaluate students' understanding of FPG. This framework by showing the levels and stages in students' concept construction of FPG help to accurately evaluate students' understanding in FPG. Rivera (2013) organized a framework that takes into account various aspects of pattern generalization. One aspect was the types of structures. The findings about structure in students' concept construction of FPG are align with Mulligan and Mitchelmore (2009) who studied learners' structural representations that were the pre structural stage, emergent, partial and the full stage of structural development.

The existence of levels between the stages in the APOS theory is expressed in a few research studies; (Arnon, 1998; Dubinsky et al., 2013; Weller et al., 2009). The levels in APOS theory can be different depending on different concepts. This study is an additive case to the aforementioned research that shows the existence levels between stages in FPG concept. This study shows that there is a huge gap at the passage from the step of writing an algebraic relation to manipulating the relation. This shows that students can use symbols to illustrate the relationship, but that it is not surely a sign of understanding symbols and their effectiveness. Instead, they use symbols in a superficial and meaningless way. This gap is the reason to explore the process of this transmission and probable existence of levels before the object stage. This study shows similar gap before the action stage and the process stage.

The obtained data confirmed the existence of these levels. We used Mulligan and Mitchelmore's (2009) levels for representation of structure to classify students' responses in different levels before stages. The results of this study show that there are emergent and partial action levels before the action stage. Also, there are pre, emergent and partial process/ object levels before process/object stage.

This research has provided a framework that can be used for 1) Identifying levels and stages of students in FPG, 2) justifying the existence of levels between stages at APOS theory, and 3) helping teachers evaluate students' understanding in FPG and to design FPG tasks.

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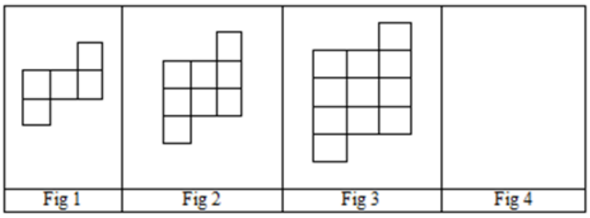
APPENDICES

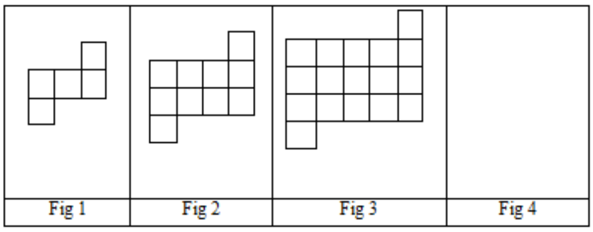
Appendix 1: Instrument 1 (Initial test: tasks 1-4, task 5, task 6)

Task 1: One variable linear figural pattern



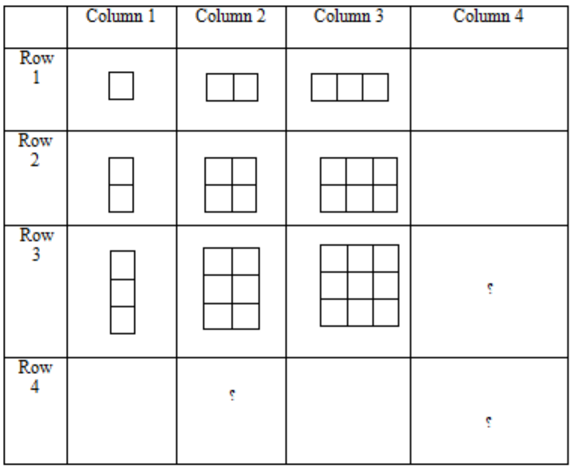
1- Draw figure 4 in the question sheet.
2- Describe how do you draw figure 5?
3- Which figure number contains 20 squares?
4- How many squares are there in figure 75?
5- How many squares are there in figure n?
6- Which figure number contains 302 squares?

Task 2: One variable quadratic figural pattern



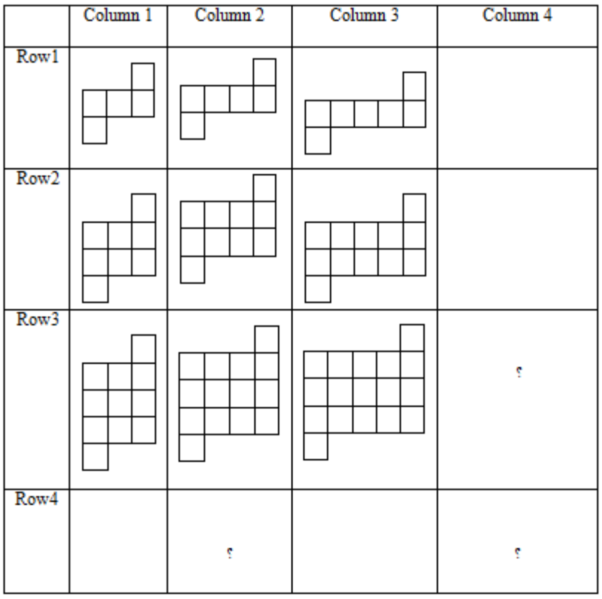
7- Draw figure 4 in the question sheet.
8- Describe how do you draw figure 5?
9- Which figure number contains 50 squares?
10- How many squares are there in figure 75?
11- How many squares are there in figure n?
12- Which figure number contains 962 squares?

Task 3: Two variable bi-linear figural pattern



13- Draw figures in places that have "?" sign, in the question sheet.
14- Describe how draw the figure in row 3 and column 4.
15- If figure in row 4, contains 36 squares, in which column is located?
16- How many squares are there in figure row 30 and column 40?
17- How many squares are there in figure row n and column m?
18- If figure in row 40, contains 480 squares, in which column is located?

Task 4: Two-variable figural pattern



19- Draw figures in places that have "?" sign, in the question sheet.
20- Describe how draw the figure in row 4 and column 2.
21- If figure in row 4, contains 34 squares, in which column is located?
22- How many squares are there in figure row 40 and column 50?
23- How many squares are there in figure row n and column m?
24- If figure in column 18, contains 762 squares, in which row is located?

Task 5: Instrument for assessing and identifying Object stage

25. Design figural pattern for $3n + 1$. For which n, the pattern will have 52 squares?
26. In the given figural pattern, shows that two obtained relations are equivalence.
27. In the given figural pattern, how the pattern changes, at the time that the numbers of figure triplicate?
28. In the given figural pattern, between witch values of n, the squares in pattern change between 55 to185?

Task 6: Instrument for assessing and identifying Schema stage

29. What is the difference between quadratic figural pattern and linear figural pattern?
30. What are the characteristics of a two-variable figural pattern that are distinct from one variable figural pattern?