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Cohomology of infinity-categories

Trabajo Fin de Máster
Doble Máster MAES-MUM

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Resumen

En este trabajo definimos una teoría de cohomología para conjuntos simpliciales entendidos como ∞-categorías. Tras su definición, estudiamos las propiedades elementales de la cohomología y especialmente su relación con cierta generalización de los espacios de Eilenberg-MacLane.

Abstract

In this memoir we define a cohomology theory for simplicial sets thought of as ∞-categories. After its definition, we study the elementary properties of the cohomology and especially its relation with a certain generalization of Eilenberg-MacLane spaces.

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Chapter 1

Introduction

1.1 Motivation

> Mathematical concepts [...] are brought into being by a series of successive abstractions and generalizations, each resting on a combination of experience with preceding abstract concepts.
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> *A. D. Aleksandrov*, "Mathematics: Its Content, Methods and Meaning"

During the first half of the twentieth century, the blossoming of algebraic topology was marked by the development of homotopy, homology and cohomology. These functors-to-be were the motivation behind the birth of category theory, which in turn provided the language and necessary abstraction for their emancipation from algebraic topology.

For instance, homology and cohomology engendered homological algebra, studying the properties of the involved functors and of (co)chain complexes. The emancipation from algebraic topology also brought with it the expansion of the scope of these tools, not being confined to classifying topological spaces, instead being applied to various geometric and algebraic structures. The example we are more interested in is the development of a cohomology theory for small categories.

Meanwhile, category theory became a consolidated subject and thus began a process of abstraction of its own to higher category theory, mainly inspired precisely by homotopy theory and its requirements. The category of categories is a motivating example for n-categories as it not only has objects and morphisms, it also naturally has 2-morphisms between its (1-)morphisms, hence constituting a first example of a 2-category. This process can be repeated with 3-morphisms between 2-morphisms, 4-morphisms between 3-morphisms... up to ∞-categories, where there exist k-morphisms for each $k \geq 1$.

However, there needs to be some conditions on the associativity and identity of these k-morphisms. The choice of these conditions is not trivial and, in order to account for some important motivating examples, the notion of associativity needs to be weakened from equality to a more subtle notion of equivalence. The vagueness of this definition made possible the appearance of multiple models for the notion of ∞-category. Joyal defined a homotopy theory (in technical terms, a model category structure) on simplicial sets which has proven to be the most successful model for the theory of infinity-categories.

Now, as ∞-category theory has grown and become more fruitful, it needs more tools. One of the goals of any theory is to classify its objects in some sense, and one tool that has proven useful in this task is cohomology. That is what this memoir aims to do, develop a cohomology theory for ∞-categories. This cohomology generalizes the cohomology of small categories presented by Baues and Wirsching in [1].

One possible application for this cohomology is to construct a Postnikov system interpolating between a simplicial set and its fundamental category. Every step of this interpolation would be classified by a cohomology class, analogously to the way in which Postnikov systems for topological spaces are constructed. This application exceeds the scope of this memoir.

In chapter 2 of this memoir we explain the basic ingredients needed to develop our theory. We begin recalling some results from category theory, then present simplicial sets and their relation to categories, in order to define quasicategories. The last two sections of the chapter present the cohomology of small categories. In chapter 3 we define the cohomology of ∞-categories and study its properties. Then we study the relation of the cohomology with a generalization of Eilenberg-MacLane spaces and finish proving that this generalization is also a quasicategory.

1.2 Notation

Although we will try to clarify the notation used in each section and result, there are some conventions assumed throughout this memoir that we set now.

In a sequence of consecutive natural numbers with one of them missing we will denote the one missing with a circumflex, as in:

$$(0,1,\dots,i-1,i+1,\dots,n-1,n) \;=\; (0,1,\dots,\widehat{i},\dots,n-1,n).$$

When talking about categories, an unspecified one will be denoted by $\mathbf{C}$, its objects $\mathrm{Obj}(\mathbf{C})$ and its morphisms $\mathrm{Mor}(\mathbf{C})$.

Morphisms between a pair of objects A, B of a category $\mathbf{C}$ will be denoted by $\mathrm{Mor}_{\mathbf{C}}(A,B)$. Although in some cases we will drop the subscript if the context category is clear or irrelevant.

The opposite category of $\mathbf{C}$ will be denoted by $\mathbf{C}^{\mathrm{op}}$.

Any functor will be assumed to be covariant.

The identity morphism for an object C will be denoted by id_C, although in some cases we will drop the subscript if the context category is clear or irrelevant.

Finally, we are going to use some known categories that will be denoted with boldface letters:

- **Set** is the category of sets and maps.
- **Ab** is the category of abelian groups and group homomorphisms.
- **Cat** is the category of small categories and functors.

Chapter 2

Preliminaries

In this chapter we are going to list some known results and definitions that will be used throughout our memoir.

2.1 Category theory

In order to talk about ∞-categories we are obviously going to need to use (1-)category theory. Although we will take most of it as known, there are a few definitions that we are going to reference explicitly. These results mainly follow [2] and their proofs are available there.

Definition 2.1.1 (Natural isomorphism)**.** A natural transformation $\alpha\colon F \to G$ will be called a natural isomorphism if for every object C in the source of F and G, the component $\alpha_C\colon F(C) \to G(C)$ is an isomorphism. We will denote it by $\alpha\colon F \cong G$.

Definition 2.1.2 (Equivalence of categories)**.** An equivalence of two categories $\mathbf{C}$ and $\mathbf{C}'$ consists of two functors together with two natural isomorphisms:

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "F"] & \mathbf{C}', \arrow[l, shift left, "G"] \end{tikzcd}$$

$$\eta\colon \mathrm{id}_{\mathbf{C}} \cong GF,$$
$$\epsilon\colon FG \cong \mathrm{id}_{\mathbf{C}'}.$$

Definition 2.1.3 (Adjoint functors)**.** An adjunction between two categories $\mathbf{C}$ and $\mathbf{C}'$ is a pair of functors $F\colon \mathbf{C} \to \mathbf{C}'$ and $G\colon \mathbf{C}' \to \mathbf{C}$ and a natural bijection for each $C \in \mathrm{Obj}(\mathbf{C})$ and $C' \in \mathrm{Obj}(\mathbf{C}')$:

$$\mathrm{Mor}_{\mathbf{C}'}(F(C), C') \cong \mathrm{Mor}_{\mathbf{C}}(C, G(C')).$$

The functor F is called the left adjoint and G the right adjoint.

Lemma 2.1.4 (Yoneda lemma)**.** Given a category $\mathbf{C}$ and a functor $F\colon \mathbf{C} \to \mathbf{Set}$. For any object A in $\mathbf{C}$, there exists a bijection:

$$\mathrm{Mor}_{\mathbf{Set}}(\mathrm{Mor}_{\mathbf{C}}(A, -), F) \cong F(A),$$

associating to each natural transformation $\alpha\colon \mathrm{Mor}_{\mathbf{C}}(A, -) \to F$ the element $\alpha_A(\mathrm{id}_A)$ of the set $F(A)$.

The last result of this section explains the construction of a category from a given partially ordered set (poset), which we will use later:

Definition 2.1.5 (Category of a poset)**.** Given a poset S, it can be viewed as a category (which we will also denote by S) with:

- *Objects:* elements of the set S.
- *Morphisms:* there exists a unique morphism $i \to j$ if and only if $i \geq j$, for any $i, j \in S$.

Any order-preserving map between posets, $f\colon S \to S'$, can also be viewed as a functor between their associated categories.

2.2 Simplicial objects

After category theory, the other ingredient we need to define ∞-categories are simplicial sets. In this section we list some basic definitions for them.

Definition 2.2.1 (Simplex category)**.** The simplex category, denoted by $\mathbf{\Delta}$, is given by:

- *Objects:* finite totally ordered sets $[n] = \{0 < 1 < \ldots < n\}$ for $n \geq 0$.
- *Morphisms:* weakly order-preserving maps between those sets. More explicitly, given a morphism $\alpha\colon [m] \to [n]$, for each $0 \leq i < j \leq m$, it satisfies $0 \leq \alpha(i) \leq \alpha(j) \leq n$.

There are some special morphisms in $\mathbf{\Delta}$, the cofaces and codegeneracies, denoted respectively by d^i and s^i for each $n \geq 0$ and for each $0 \leq i \leq n$.

$$\begin{aligned} d^i &\colon [n-1] \to [n], \\ d^i(j) &= \begin{cases} j & \text{if} \quad j < i, \\ j+1 & \text{if} \quad j \geq i, \end{cases} \\ s^i &\colon [n+1] \to [n], \\ s^i(j) &= \begin{cases} j & \text{if} \quad j \leq i, \\ j-1 & \text{if} \quad j > i. \end{cases} \end{aligned} \tag{2.2.2}$$

Any morphism $\alpha\colon [m] \to [n]$ in $\mathbf{\Delta}$ can be factorized as a composition of cofaces and codegeneracies, so we will usually only consider these instead of all morphisms.

The objects of $\mathbf{\Delta}$ can also be viewed as categories, as in Definition 2.1.5, and their morphisms as functors.

Definition 2.2.3 (Simplicial object)**.** A simplicial object in a category $\mathbf{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$.

A particular type of simplicial objects are simplicial sets, which we can describe explicitly:

Definition 2.2.4 (Simplicial set)**.** A simplicial set X is a family of sets and maps between them:

$$\begin{tikzcd} \cdots & X_n \arrow[r, shift left, "d_0,\ldots,d_n"] & X_{n-1} \arrow[l, shift left, "s_0,\ldots,s_{n-1}"] & \cdots & X_2 \arrow[r, shift left, "d_0,d_1,d_2"] & X_1 \arrow[l, shift left, "s_0,s_1"] \arrow[r, shift left, "d_0,d_1"] & X_0. \arrow[l, shift left, "s_0"] \end{tikzcd}$$

More precisely, we have the following ingredients:

- Sets X_n for each $n \geq 0$, called sets of n-simplices.
- Face and degeneracy maps $d_i \colon X_n \to X_{n-1}$ and $s_i \colon X_n \to X_{n+1}$, for $0 \leq i \leq n$, satisfying the following conditions, called *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \text{ if } i < j, \\ d_i s_j &= s_{j-1} d_i \text{ if } i < j, \\ d_j s_j &= d_{j+1} s_j = \mathrm{id}, \\ d_i s_j &= s_j d_{i-1} \text{ if } i > j+1, \\ s_i s_j &= s_{j+1} s_i \text{ if } i \leq j. \end{aligned} \tag{2.2.5}$$

Considering Definition 2.2.3, this construction defines a functor $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ with $X([n]) = X_n$, $X(d^i) = d_i$ and $X(s^i) = s_i$.

Definition 2.2.6 (Simplicial map)**.** Given two simplicial sets X and Y, a simplicial map between them, $f \colon X \to Y$, is a collection of maps between sets $f_n \colon X_n \to Y_n$ for $n \geq 0$, satisfying the following condition for any $\alpha \colon [m] \to [n]$:

$$f_m \circ X(\alpha) = Y(\alpha) \circ f_n.$$

where we are considering X and Y as functors.

We know that any morphism $\alpha \colon [m] \to [n]$ can be factorized as a composition of cofaces and codegeneracies, thus for $f \colon X \to Y$ to be a simplicial map it suffices that:

$$\begin{aligned} f_n d_i &= d_i f_{n+1}, \text{ for each } 0 \leq i \leq n+1, \\ f_n s_i &= s_i f_{n-1}, \text{ for each } 0 \leq i \leq n-1. \end{aligned}$$

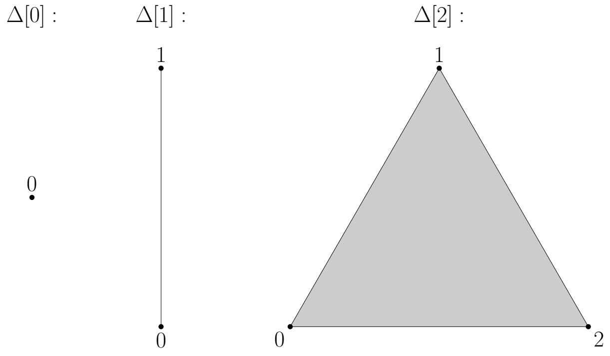
From now on, we will not use the subscript in f_n, instead we will distinguish each of the maps by context.

We will denote the category of simplicial sets and simplicial maps by **sSet**.

There is a particular family of simplicial sets which we are going to use extensively in this memoir:

Definition 2.2.7 (n-simplex)**.** The standard n-simplex, which we will refer to as n-simplex, is the simplicial set $\Delta[n] \coloneqq \mathrm{Mor}_{\mathbf{\Delta}}(-, [n])$.

The set of n-simplices of $\Delta[n]$, $\Delta[n]_n$, is defined as $\mathrm{Mor}_{\mathbf{\Delta}}([n], [n])$ and thus it has a distinguished element $\mathrm{id}_{[n]}$. The element $d_i(\mathrm{id}_{[n]}) \in \Delta[n]_{n-1}$ will be called the i-th face of the n-simplex. We can depict the first few simplices as:

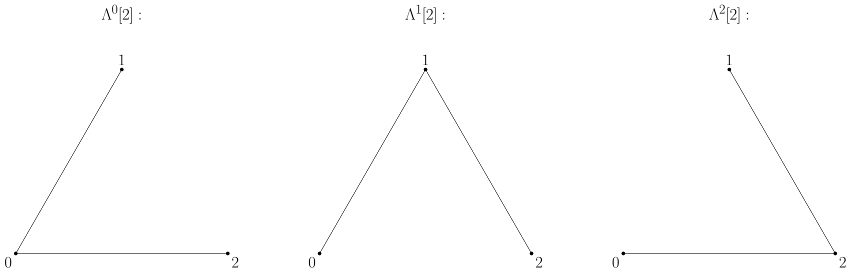


Definition 2.2.8 (Horns of a simplex)**.** For $n \geq 1$ and $0 \leq i \leq n$, the i-horn of the n-simplex, $\Lambda^i[n]$, is, intuitively, the sub-simplicial set of $\Delta[n]$ given by the union of its faces except the i-th one. More explicitly:

$$\Lambda^i[n]_m = \{\alpha\colon [m] \to [n] \mid ([n] \setminus \{i\}) \not\subset \alpha([m])\}.$$

The i-horns for $0 < i < n$ will be called inner horns.

For example, the 2-simplex has three horns, but only one of them, $\Lambda^1[2]$, is an inner horn:



Corollary 2.2.9. For any simplicial set X, applying the Yoneda Lemma we obtain that its n-simplices are in bijection with the simplicial maps from the standard n-simplex to X:

$$X_n \cong \mathrm{Mor}_{\mathbf{sSet}}(\Delta[n], X).$$

This means that any $a \in X_n$ can be thought of as a simplicial map $a\colon \Delta[n] \to X$, satisfying precisely $a(\mathrm{id}_{[n]}) = a \in X_n$. We will denote both elements simply by a and will distinguish between them only by context.

2.3 Simplicial sets and categories

There are two important functors between categories and simplicial sets which are going to play a central role in this memoir.

Definition 2.3.1 (Nerve of a category)**.** Given a small category $\mathbf{C}$, its nerve $\mathrm{N}_\bullet(\mathbf{C})$ is a simplicial set with $\mathrm{N}_n(\mathbf{C})$ defined as the set of functors from $[n]$ to $\mathbf{C}$:

$$\mathrm{N}_n(\mathbf{C}) = \mathrm{Mor}_{\mathbf{Cat}}([n], \mathbf{C}).$$

Given a coface map of $\mathbf{\Delta}$, $d^i\colon [n-1] \to [n]$, it induces face maps on the nerve, $d_i\colon \mathrm{N}_n(\mathbf{C}) \longrightarrow \mathrm{N}_{n-1}(\mathbf{C})$, defined by precomposition with d^i:

$$\begin{tikzcd}
{[n-1]} \arrow[r, "d^i"] \arrow[rr, bend right, "d_i(f)"'] & {[n]} \arrow[r, "f"] & \mathbf{C}.
\end{tikzcd}$$

Codegeneracies $s^i\colon [n] \to [n-1]$ analogously induce degeneracies $s_i\colon \mathrm{N}_{n-1}(\mathbf{C}) \to \mathrm{N}_n(\mathbf{C})$.

An element of $\mathrm{N}_n(\mathbf{C})$ (a functor $[n] \to \mathbf{C}$) can be viewed as an n-tuple of composable morphisms in $\mathbf{C}$:

$$C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0.$$

We will denote such an element by $(f_1, f_2, \dots, f_n) \in \mathrm{N}_n(\mathbf{C})$. Applying the face and degeneracy maps yields:

$$\begin{aligned} d_0(f_1, \dots, f_n) &= (f_2, \dots, f_n), \\ d_i(f_1, \dots, f_n) &= (f_1, \dots, f_i f_{i+1}, \dots, f_n), \text{ for } 0 < i < n, \\ d_n(f_1, \dots, f_n) &= (f_1, \dots, f_{n-1}), \\ s_i(f_1, \dots, f_n) &= (f_1, \dots, f_i, \mathrm{id}_{C_i}, f_{i+1}, \dots, f_n), \text{ for } 0 \leq i \leq n. \end{aligned} \tag{2.3.2}$$

There are two other related identities that will be useful later:

$$\begin{aligned} d_1^{n-1}(f_1, \dots, f_n) &= f_1 \circ \dots \circ f_n, \\ d_0^{i-1} d_{i+1}^{n-i}(f_1, \dots, f_n) &= f_i. \end{aligned} \tag{2.3.3}$$

Another convenient fact is that the nerve of the category $[n]$ is the standard n-simplex:

$$\mathrm{N}_m([n]) = \mathrm{Mor}_{\mathbf{Cat}}([m], [n]) = \mathrm{Mor}_{\mathbf{\Delta}}([m], [n]) = \Delta[n]_m.$$

We claim that the nerve construction defines a functor $\mathbf{Cat} \to \mathbf{sSet}$, for that we need to define how it acts on morphisms:

Definition 2.3.4. Given a functor $F\colon \mathbf{C} \to \mathbf{C}'$, it induces a simplicial map $\mathrm{N}(F)\colon \mathrm{N}(\mathbf{C}) \to \mathrm{N}(\mathbf{C}')$ sending $(f_1, \dots, f_n) \in \mathrm{N}_n(\mathbf{C})$ to:

$$\mathrm{N}(F)(f_1, \dots, f_n) = (F(f_1), \dots, F(f_n)) \in \mathrm{N}_n(\mathbf{C}').$$

Now, we define the other functor, from simplicial sets to categories:

Definition 2.3.5 (Fundamental category)**.** Given a simplicial set X, its fundamental category, $\tau_1 X$, is defined as follows:

- *Objects:* the 0-simplices of X, that is, the elements of the set X_0.
- *Morphisms:* the morphisms of $\tau_1 X$ are generated by the elements of X_1 regarded as morphisms $a\colon d_1 a \to d_0 a$ for $a \in X_1$, modulo the relations $d_1 a \sim d_2 a \circ d_0 a$ for $a \in X_2$ and $s_0 a \sim \mathrm{id}_a$ for $a \in X_0$. Given an element $a \in X_1$, we will denote its class by $\{a\}$.

This shows how the functor $\tau_1\colon \mathbf{sSet} \to \mathbf{Cat}$ behaves on objects, on morphisms we have:

Definition 2.3.6. Given a simplicial map $f\colon X \to Y$, it induces a functor $\tau_1 f\colon \tau_1 X \to \tau_1 Y$ defined:

- *On objects:* given $a \in \mathrm{Obj}(\tau_1 X) = X_0$, $\tau_1 f(a) = f(a) \in Y_0 = \mathrm{Obj}(\tau_1 Y)$.
- *On morphisms:* given $\{a\}\colon d_1 a \to d_0 a$, we define $\tau_1 f(\{a\}) = \{f(a)\}\colon d_1 f(a) \to d_0 f(a)$, .

One fact about these functors is that they have a nice relationship between them:

Lemma 2.3.7. The nerve functor is right adjoint to the fundamental category functor:

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, shift left, "\tau_1"] & \mathbf{Cat}, \arrow[l, shift left, "\mathrm{N}"] \end{tikzcd}$$

which means there is a natural bijection between the sets of morphisms $\mathrm{Mor}_{\mathbf{Cat}}(\tau_1 X, \mathbf{C}) \cong \mathrm{Mor}_{\mathbf{sSet}}(X, \mathrm{N}\mathbf{C})$ for each simplicial set X and category $\mathbf{C}$.

This is proven, for example, in [3, Corollary 004N].

This means that the identity functor $\tau_1 X \to \tau_1 X$ induces a unique simplicial morphism:

$$\Psi\colon X \longrightarrow \mathrm{N}\tau_1 X. \tag{2.3.8}$$

More explicitly, the image by Ψ of an element $a \in X_n$ is defined as:

$$C_n \xrightarrow{\{a_n\}} C_{n-1} \xrightarrow{\{a_{n-1}\}} \cdots \xrightarrow{\{a_2\}} C_1 \xrightarrow{\{a_1\}} C_0,$$

with $\{a_i\} = \{d_0^{i-1} d_{i+1}^{n-i} a\}$ and $C_i = d_0^i d_{i+1}^{n-i} a$ for $1 \leq i \leq n$ and $C_0 = d_1^n a$.

Analogously, the identity $\mathrm{N}\mathbf{C} \to \mathrm{N}\mathbf{C}$ induces a canonical functor $\tau_1 \mathrm{N}\mathbf{C} \to \mathbf{C}$, but in this case it is an identity. This result can be found in [3, Remark 00HG].

2.4 ∞-category theory

Having explained categories and simplicial sets, now we can present and define our model for ∞-categories: quasicategories. We will previously define the related concept of Kan complex.

Definition 2.4.1 (Kan complex)**.** A simplicial set X is a Kan complex if for any map $\Lambda^i[n] \to X$, and taking the canonical inclusion $\Lambda^i[n] \hookrightarrow \Delta[n]$, there exists a morphism from the n-simplex to X making the following diagram commute:

$$\begin{tikzcd}
\Lambda^i[n] \arrow[r] \arrow[d, hook] & X \\
\Delta[n] \arrow[ur, dashed] &
\end{tikzcd}$$

In this case the morphism $\Delta[n] \to X$ will be called a filler morphism.

Definition 2.4.2 (Quasicategory)**.** A quasicategory is a simplicial set X satisfying the Kan condition (having a filler morphism) for the inner horns, $0 < i < n$.

$$\begin{tikzcd}
\Lambda^i[n] \arrow[r] \arrow[d, hook] & X \\
\Delta[n] \arrow[ur, dashed] &
\end{tikzcd} \qquad \text{for } 0 < i < n.$$

After defining quasicategories, we want a way to define an analogous concept to the equivalence of 1-categories, but in this case we will need some previous notation:

Definition 2.4.3. Given two simplicial sets X and Y, we define the simplicial set of morphisms between them, X^Y, as:

$$X_n^Y = \mathrm{Mor}_{\boldsymbol{\Delta}}(Y \times \Delta[n], X).$$

With faces, $d_i\colon X_n^Y \to X_{n-1}^Y$, defined for $\gamma \in X_n^Y$ as:

$$\begin{tikzcd}
Y \times \Delta[n-1] \arrow[r, "\mathrm{id} \times d^i"] \arrow[rr, bend right, "d_i\gamma"'] & Y \times \Delta[n] \arrow[r, "\gamma"] & X.
\end{tikzcd}$$

And degeneracies defined analogously.

Definition 2.4.4. Given three simplicial sets X, Y, Z and a simplicial map $f\colon X \to Y$, it induces a map $f^*\colon Z^Y \to Z^X$ defined as:

$$\begin{tikzcd}
Y \times \Delta[n] \arrow[dd, "\gamma"', ""{name=A}] & & X \times \Delta[n] \arrow[dd, "f^*\gamma", ""'{name=B}] \arrow[dr, dotted, "f\times\mathrm{id}"] & \\
 & & & Y \times \Delta[n]. \arrow[dl, dotted, "\gamma"] \\
Z & & Z &
\arrow[from=A, to=B, mapsto]
\end{tikzcd}$$

Definition 2.4.5 (Quasiequivalence)**.** Given two simplicial sets X and Y and a simplicial map between them, $f\colon X \to Y$. We say f is a quasiequivalence if for every quasicategory Z the induced map $\tau_1(f^*)$ is an equivalence of 1-categories:

$$\tau_1(f^*)\colon \tau_1(Z^Y) \longrightarrow \tau_1(Z^X).$$

2.5 Natural systems

The cohomology of ∞-categories we are going to develop in later sections will be defined with natural systems as its coefficients, a concept which was introduced in [1].

Natural systems are not directly defined on a category, instead, they use a certain endofunctor on the category of small categories, which we define now:

Definition 2.5.1 (Factorization category)**.** Given a small category $\mathbf{C}$, its factorization category, $\mathcal{F}\mathbf{C}$, is defined as follows:

- *Objects:* morphisms of $\mathbf{C}$.
- *Morphisms:* a morphism $(h,k)\colon f \to g$ is a commutative diagram:

$$\begin{tikzcd}
\bullet \arrow[r, "h"] & \bullet \\
\bullet \arrow[u, "f"] & \bullet \arrow[l, "k"] \arrow[u, "g"']
\end{tikzcd},$$

  i.e. such that $g = hfk$. Composition is defined as $(h',k')(h,k) = (hh', k'k)$.

This construction induces a functor $\mathcal{F}\colon \mathbf{Cat} \to \mathbf{Cat}$, defined on objects in the obvious way. A functor $F\colon \mathbf{C} \to \mathbf{C}'$ defines another one:

$$\mathcal{F}(F)\colon \mathcal{F}\mathbf{C} \longrightarrow \mathcal{F}\mathbf{C}'.$$

This functor $\mathcal{F}(F)$ is defined on objects (which are morphisms of $\mathbf{C}$) as $\mathcal{F}(F)(f) = F(f)$ and on morphisms as $\mathcal{F}(F)(h,k) = (F(h), F(k))$.

Now, we can define what a natural system is.

Definition 2.5.2 (Natural system)**.** Given a small category $\mathbf{C}$, a natural system (of abelian groups) on $\mathbf{C}$ is a functor $D\colon \mathcal{F}\mathbf{C} \to \mathbf{Ab}$. Each morphism $(h,k)\colon f \to g$ in $\mathcal{F}\mathbf{C}$ induces a homomorphism:

$$D(h,k)\colon D(f) \longrightarrow D(hfk) = D(g).$$

Given a natural system on $\mathbf{C}$, $D\colon \mathcal{F}\mathbf{C} \to \mathbf{Ab}$, any functor $F\colon \mathbf{C}' \to \mathbf{C}$ induces a natural system on $\mathbf{C}'$, denoted by $F^*(D)$:

$$\begin{tikzcd}
\mathcal{F}\mathbf{C}' \arrow[r, "\mathcal{F}(F)"] \arrow[rr, bend right, "F^*(D)"'] & \mathcal{F}\mathbf{C} \arrow[r, "D"] & \mathbf{Ab},
\end{tikzcd} \tag{2.5.3}$$

defined simply by $F^*(D)(f) = D(F(f))$ and $F^*(D)(h,k) = D(F(h),F(k))$.

To finish this section, we define the category of all natural systems. This category is an adaptation of the one presented in [1], although centered on ∞-categories instead of ordinary ones. Nevertheless, we will use the same notation as Baues and Wirsching.

Definition 2.5.4 (Category of natural systems)**.** The category **Nat** of natural systems on ∞-categories is defined with:

- *Objects:* pairs (X,D), where X is a simplicial set and D a natural system on $\tau_1 X$.
- *Morphisms:* $(\lambda,\mu)\colon (X,D) \to (X',D')$ where $\lambda\colon X' \to X$ is a simplicial map and $\mu\colon (\tau_1\lambda)^*D \to D'$ is a natural transformation.

With composition for $(\lambda,\mu)\colon (X,D) \to (X',D')$ and $(\lambda',\mu')\colon (X',D') \to (X'',D'')$ defined as:

$$(\lambda',\mu') \circ (\lambda,\mu) = (\lambda\lambda', \mu' \circ (\tau_1\lambda')^*\mu),$$

$$X'' \xrightarrow{\lambda'} X' \xrightarrow{\lambda} X,$$

$$(\tau_1\lambda')^*(\tau_1\lambda)^*D \xrightarrow{(\tau_1\lambda')^*\mu} (\tau_1\lambda')^*D' \xrightarrow{\mu'} D''.$$

This means $(\lambda,\mu) = (\mathrm{id},\mu)(\lambda,\mathrm{id})\colon (X,D) \to (X',D')$, where $(\lambda,\mathrm{id})\colon (X,D) \to (X',(\tau_1\lambda)^*D)$ and $(\mathrm{id},\mu)\colon (X',(\tau_1\lambda)^*D) \to (X',D')$.

2.6 Baues-Wirching's cohomology

We recall the cohomology of small categories defined by Baues and Wirsching to compare it later with our own construction:

Definition 2.6.1 (Baues-Wirsching's cochains)**.** Given a small category $\mathbf{C}$ and a natural system $D\colon \mathcal{F}\mathbf{C} \to \mathbf{Ab}$, the n-th Baues-Wirsching cochain group of $\mathbf{C}$ with coefficients in D, $F^n_{BW}(\mathbf{C},D)$, is the abelian group of maps:

$$f\colon \mathrm{N}_n(\mathbf{C}) \longrightarrow \bigsqcup_{g\in\mathrm{Mor}(\mathbf{C})} D(g), \text{ with } f(g_1,\ldots,g_n) \in D(g_1 \circ \cdots \circ g_n),$$

for $n > 0$. For $n = 0$, $F^0_{BW}(\mathbf{C},D)$ is the abelian group of maps:

$$f\colon \mathrm{N}_0(\mathbf{C}) \longrightarrow \bigsqcup_{A\in\mathrm{Obj}(\mathbf{C})} D(\mathrm{id}_A), \text{ with } f(A) \in D(\mathrm{id}_A).$$

The coboundary between cochains, $\partial_{BW}\colon F^{n-1}_{BW}(\mathbf{C}, D) \to F^{n}_{BW}(\mathbf{C}, D)$ is defined as:

$$\begin{aligned}\partial_{BW} f(a_1, \ldots, a_n) = & D(a_1, \mathrm{id}) f(a_2, \ldots, a_n) \\ & + \sum_{i=1}^{n-1} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\ & + (-1)^n D(\mathrm{id}, a_n) f(a_1, \ldots, a_{n-1})\end{aligned} \tag{2.6.2}$$

for $n > 1$. For $n = 1$, with $a\colon A \to B$:

$$\partial_{BW} f(a) = D(a, \mathrm{id}) f(A) - D(\mathrm{id}, a) f(B). \tag{2.6.3}$$

We will denote the n-cocycles by $Z^n_{BW}(\mathbf{C}, D)$.

Definition 2.6.4 (Baues-Wirsching's cohomology)**.** Given a small category $\mathbf{C}$ and a natural system on $\mathbf{C}$, D. We define the cohomology of $\mathbf{C}$ with coefficients in D, $H^n_{BW}(\mathbf{C}, D)$, as the cohomology of the cochain complex $F^n_{BW}(\mathbf{C}, D)$ with the coboundary ∂_{BW}.

Chapter 3

Cohomology of ∞-categories

With all the previous prerequisites laid out we can finally define the cohomology of an ∞-category and its elementary properties. To simplify the writing from now on, unless we say otherwise, we assume X is a simplicial set and D is a natural system on $\tau_1 X$.

3.1 Definition and basic properties

Definition 3.1.1 (Cochains)**.** The n-cochains of X with coefficients in D, $F^n(X, D)$, are the abelian group of maps:

$$f\colon X_n \longrightarrow \coprod_{g \in \mathrm{Mor}(\tau_1 X)} D(\{g\}), \text{ such that } f(a) \in D(\{d_1^{n-1} a\}),$$

for $n > 0$. For $n = 0$, $F^0(X, D)$ is the abelian group of maps:

$$f\colon X_0 \longrightarrow \coprod_{a \in \mathrm{Obj}(\tau_1 X)} D(\{\mathrm{id}_a\}), \text{ such that } f(a) \in D(\{\mathrm{id}_a\}).$$

Its coboundary $\partial\colon F^{n-1}(X, D) \to F^n(X, D)$, is defined for $a \in X_n$ as:

$$\partial f(a) = D(\{d_2^{n-1} a\}, \mathrm{id}) f(d_0 a) + \sum_{i=1}^{n-1} (-1)^i f(d_i a) + (-1)^n D(\mathrm{id}, \{d_0^{n-1} a\}) f(d_n a) \tag{3.1.2}$$

for $n > 1$. Lastly, for $n = 1$ the coboundary is defined as:

$$\partial f(a) = D(\{a\}, \mathrm{id}) f(d_0 a) - D(\mathrm{id}, \{a\}) f(d_1 a). \tag{3.1.3}$$

We will also denote the n-cocycles by $Z^n(X, D)$.

Having defined the cochains, the next step is to check that they are correctly defined and work as expected:

Lemma 3.1.4. The coboundary ∂ is well defined.

Proof. We must show that the coboundary is compatible with the natural system, that is, we want to check that $\partial f(a)$ actually belongs to $D(\{d_1^{n-1} a\})$.

We begin by checking the case $n > 1$. In the first summand of (3.1.2) we have $f(d_0 a) \in D(\{d_1^{n-2} d_0 a\})$ and thus:

$$D(\{d_2^{n-1} a\}, \mathrm{id}) f(d_0 a) \in D(\{d_2^{n-1} a \circ d_1^{n-2} d_0 a\}).$$

However, using the simplicial identities and the relations of $\mathrm{Mor}(\tau_1 X)$ we get:

$$\{d_2^{n-1}a \circ d_1^{n-2}d_0a\} = \{d_2d_2^{n-2}a \circ d_0d_2^{n-2}a\} = \{d_1d_2^{n-2}a\} = \{d_1^{n-1}a\},$$

where we are using the relation $d_1a \sim d_2a \circ d_0a$, $a \in X_2$, for the 2-simplex $d_2^{n-2}a \in X_2$.

All the terms of the summation from (3.1.2) satisfy $f(d_ia) \in D(\{d_1^{n-2}d_ia\}) = D(\{d_1^{n-1}a\})$, because $i \neq 0, n$.

The last summand is analogous to the first one:

$$\begin{gathered} D(\mathrm{id}, \{d_0^{n-1}a\})f(d_na) \in D(\{d_1^{n-2}d_na \circ d_0^{n-1}a\}) \\ \text{with } \{d_1^{n-2}d_na \circ d_0^{n-1}a\} = \{d_2d_1^{n-2}a \circ d_0d_1^{n-2}a\} = \{d_1^{n-1}a\} \end{gathered}$$

In the case $n = 1$ it is clear that $\partial f(a) \in D(\{a\})$, because $a \in X_1$, which implies $f(d_0a) \in D(\{\mathrm{id}_{d_0a}\})$ and $f(d_1a) \in D(\{\mathrm{id}_{d_1a}\})$, so $D(\{a\}, \mathrm{id})f(d_0a)$ and $D(\mathrm{id}, \{a\})f(d_1a)$ are in $D(\{a\})$. □

Lemma 3.1.5. The coboundary ∂, satisfies the cochain complex condition, $\partial^2 = 0$.

Proof. For $f \in F^{n-2}(X, D)$ and $a \in X_n$:

$$\begin{aligned}
(\partial^2 f)(a) =& D(\{d_2^{n-1}a\}, \mathrm{id})(\partial f)(d_0a) + \sum_{i=1}^{n-1}(-1)^i(\partial f)(d_ia) + (-1)^n D(\mathrm{id}, \{d_0^{n-1}a\})(\partial f)(d_na) = \\
=& D(\{d_2^{n-1}a\}, \mathrm{id})\Bigg(\underbrace{D(\{d_2^{n-2}d_0a\}, \mathrm{id})f(d_0^2a)}_{(1)} + \underbrace{\sum_{i=1}^{n-2}(-1)^i f(d_id_0a)}_{(2)} \\
&+ \underbrace{(-1)^{n-1}D(\mathrm{id}, \{d_0^{n-1}a\})f(d_{n-1}d_0a)}_{(3)}\Bigg) + \sum_{i=1}^{n-1}(-1)^i\Bigg(\underbrace{D(\{d_2^{n-2}d_ia\}, \mathrm{id})f(d_0d_ia)}_{(4)} \\
&+ \underbrace{\sum_{j=1}^{n-2}(-1)^j f(d_jd_ia)}_{(5)} + \underbrace{(-1)^{n-1}D(\mathrm{id}, \{d_0^{n-2}d_ia\})f(d_{n-1}d_ia)}_{(6)}\Bigg) \\
&+ (-1)^n D(\mathrm{id}, \{d_0^{n-1}a\})\Bigg(\underbrace{D(\{d_2^{n-2}d_na\}, \mathrm{id})f(d_0d_na)}_{(7)} + \underbrace{\sum_{i=1}^{n-2}(-1)^i f(d_id_na)}_{(8)} \\
&+ \underbrace{(-1)^{n-1}D(\mathrm{id}, \{d_0^{n-2}d_na\})f(d_{n-1}d_na)}_{(9)}\Bigg)
\end{aligned}$$

From this point everything cancels out using the simplicial identities, the relations of the fundamental category and the composition of the maps $D(-,-)$. First of all $d_2^{n-2}d_n = d_2^{n-1}$ and $d_{n-1}d_0 = d_0d_n$, so (3) and (7) directly cancel out. Then, looking at (1), we can use that:

$$\{d_2^{n-1}a \circ d_2^{n-2}d_0a\} = \{d_2d_3^{n-2}a \circ d_0d_3^{n-2}a\} = \{d_1d_3^{n-2}a\} = \{d_2^{n-2}d_1a\},$$

which, recalling $d_0d_1 = d_0^2$, makes it coincide with the case $i = 1$ from (4) (except the sign), cancelling out. The rest of the i's from (4) can be cancelled out with the summation (2): to do so we use $d_2^{n-2}d_i = d_2^{n-1}$ (for any $2 \leq i \leq n-1$) and taking the term $i = k$ from (2) and the term $i = k+1$ from (4) we have $d_kd_0 = d_0d_{k+1}$ for $1 \leq k \leq n-2$.

The term (5) cancels out with itself. Using the first simplicial identity we have that the cases $j = 1, i = 2$ and $j = 1, i = 1$ cancel each other out. The same can be said for $j = 1, i = 3$ and $j = 2, i = 1$ and, more generally, for $j = 1, i = k$ and $j = k-1, i = 1$, where $2 \leq k \leq n-1$. We can repeat the same argument for $j = 2, i = k$ and $j = k-1, i = 2$, $3 \leq k \leq n-1$. Analogously, we can generalize this to $j = l, i = k$ and $j = k-1, i = l$, which cancel each other out for $1 \leq l \leq n-2$ and $l+1 \leq k \leq n-1$.

The rest of the terms are (6), (8) and (9), and they can be done in the same way we have done (1), (2) and (4). First, we can cancel out (9) and the case $i = n-1$ from (6), to do so we use:

$$\{d_0^{n-2}d_na \circ d_0^{n-1}a\} = \{d_2d_0^{n-2}a \circ d_0d_0^{n-2}a\} = \{d_1d_0^{n-2}a\} = \{d_0^{n-2}d_na\}.$$

Lastly, we can cancel out the cases $i = k$ from (6) with the cases $i = k$ from (8) for $1 \leq k \leq n-2$, taking into account $d_0^{n-2}d_k = d_0^{n-1}$ and $d_kd_n = d_{n-1}d_k$ when $k \leq n-2$.

□

Now we want to check that our cochain complex extends the one developed by Baues and Wirsching, explained in Definition 2.6.1.

Proposition 3.1.6. Given a small category $\mathbf{C}$ and a natural system $D\colon \mathcal{F}\mathbf{C} \to \mathbf{Ab}$, which we can also consider as $D\colon \mathcal{F}\tau_1\mathrm{N}\mathbf{C} \to \mathbf{Ab}$, we have:

$$F^\bullet(\mathrm{N}\mathbf{C}, D) = F^\bullet_{BW}(\mathbf{C}, D).$$

Proof. As we know, $\tau_1\mathrm{N}\mathbf{C} = \mathbf{C}$, which means we do not need to consider the equivalence classes associated to the morphisms of the fundamental category.

For any $n > 0$, $F^n(\mathrm{N}\mathbf{C}, D)$ is the abelian group of maps:

$$f\colon \mathrm{N}_n\mathbf{C} \longrightarrow \bigsqcup_{g \in \mathrm{Mor}(\tau_1\mathrm{N}\mathbf{C})} D(g), \text{ satisfying } f(a) \in D(d_1^{n-1}a).$$

Using that $a \in \mathrm{N}_n(\mathbf{C})$, we can write $a = (a_1, \ldots, a_n)$ and therefore $d_1^{n-1}a = a_1 \circ \cdots \circ a_n$:

$$f\colon \mathrm{N}_n\mathbf{C} \longrightarrow \bigsqcup_{g \in \mathrm{Mor}(\mathbf{C})} D(g), \text{ satisfying } f(a_1, \ldots, a_n) \in D(a_1 \circ \cdots \circ a_n),$$

which is exactly the definition of the elements of $F^n_{BW}(\mathbf{C}, D)$.

For the case $n = 0$, $F^0(\mathrm{N}\mathbf{C}, D)$ is the abelian group of maps:

$$f\colon \mathrm{N}_0\mathbf{C} \longrightarrow \bigsqcup_{a \in \mathrm{Obj}(\tau_1\mathrm{N}\mathbf{C})} D(\mathrm{id}_a), \text{ satisfying } f(a) \in D(\mathrm{id}_a),$$

which, taking into account $\tau_1\mathrm{N}\mathbf{C} = \mathbf{C}$, coincides with the definition of $F^0_{BW}(\mathbf{C}, D)$.

We now need to check that the coboundaries are equal. For the case $n > 1$ that means comparing (2.6.2) and (3.1.2). The definition of the coboundary of $F^n(\mathrm{N}\mathbf{C}, D)$ without equivalence classes is:

$$\partial f(a) = D(d_2^{n-1}a, \mathrm{id})f(d_0a) + \sum_{i=1}^{n-1}(-1)^i f(d_ia) + (-1)^n D(\mathrm{id}, d_0^{n-1}a)f(d_na).$$

We can write more explicitly the faces of $a = (a_1, \dots, a_n)$ using the identities (2.3.2) and (2.3.3):

$$\begin{aligned}\partial f(a) =& D(a_1, \mathrm{id})f(a_2, \dots, a_n) + \sum_{i=1}^{n-1}(-1)^i f(a_1, \dots, a_ia_{i+1}, \dots, a_n) \\ &+ (-1)^n D(\mathrm{id}, a_n)f(a_1, \dots, a_{n-1}),\end{aligned}$$

which is the same definition given for ∂_{BW}.

For the case $n = 1$, working with (3.1.3) and removing equivalence classes we get:

$$\partial f(a) = D(a, \mathrm{id})f(d_0a) - D(\mathrm{id}, a)f(d_1a),$$

for $a\colon d_0a \to d_1a$, meaning ∂ coincides with the ∂_{BW} defined in (2.6.3). □

With this proposition proven, we can denote both the Baues-Wirsching's cochain complex and the cochain complex defined in Definition 3.1.1 by $F^\bullet(-,-)$.

We claim that cochains define a functor $F^n\colon \mathbf{Nat} \to \mathbf{Ab}$. We know how it behaves on objects, let us see it on morphisms and then prove its functoriality. For that, we recall that any morphism $(\lambda, \mu)\colon (X, D) \to (X', D')$ in $\mathbf{Nat}$ can be decomposed as $(\lambda, \mu) = (\mathrm{id}, \mu)(\lambda, \mathrm{id})$. We are going to study these morphisms separately:

Lemma 3.1.7. A morphism $(\lambda, \mathrm{id})\colon (X, D) \to (X', (\tau_1\lambda)^*D)$ induces a homomorphism:

$$F^n(\lambda, \mathrm{id}) := \lambda^*\colon F^n(X, D) \longrightarrow F^n(X', (\tau_1\lambda)^*D),$$

defined for $f \in F^n(X, D)$ and $a \in X'_n$ by precomposition with λ:

$$\lambda^*f(a) = f(\lambda(a)).$$

Proof. We have to check that $\lambda^*f \in F^n(X', (\tau_1\lambda)^*D)$, which means checking if $\lambda^*f(a) \in (\tau_1\lambda)^*D(\{d_1^{n-1}a\})$. We know $f \in F^n(X, D)$, so:

$$\lambda^*f(a) = f(\lambda(a)) \in D(\{d_1^{n-1}\lambda(a)\}).$$

Using the fact that λ is a simplicial map, the definition of the fundamental category functor and the construction in (2.5.3), we get:

$$D(\{d_1^{n-1}\lambda(a)\}) = D(\{\lambda(d_1^{n-1}a)\}) = D((\tau_1\lambda)(\{d_1^{n-1}a\})) = (\tau_1\lambda)^*D(\{d_1^{n-1}a\}).$$

Now, we have to prove the compatibility with the coboundary, i.e. $\partial\lambda^* = \lambda^*\partial$. Given $f \in F^{n-1}(X, D)$ and $a \in X'_n$ and using the fact that λ is a simplicial map, we have:

$$\begin{aligned}\lambda^*(\partial f)(a) =& (\partial f)(\lambda a) = D(\{d_2^{n-1}\lambda a\}, \mathrm{id})f(d_0\lambda a) + \sum_{i=1}^{n-1}(-1)^i f(d_i\lambda a) + (-1)^n D(\mathrm{id}, \{d_0^{n-1}\lambda a\})f(d_n\lambda a) = \\ =& D(\{\lambda d_2^{n-1}a\}, \mathrm{id})f(\lambda d_0a) + \sum_{i=1}^{n-1}(-1)^i f(\lambda d_ia) + (-1)^n D(\mathrm{id}, \{\lambda d_0^{n-1}a\})f(\lambda d_na).\end{aligned}$$

Focusing on the first term and using the definitions of $\lambda^* f$, $\tau_1\lambda$ and $(\tau_1\lambda)^* D$ we have:

$$D(\{\lambda d_2^{n-1}a\}, \mathrm{id})f(\lambda d_0 a) = D((\tau_1\lambda)(\{d_2^{n-1}a\}), \mathrm{id})f(\lambda d_0 a) = (\tau_1\lambda)^* D(\{d_2^{n-1}a\}, \mathrm{id})(\lambda^* f)(d_0 a).$$

Then, doing analogous calculations for each term in the sum, we arrive at:

$$\begin{aligned}\lambda^*(\partial f)(a) =&(\tau_1\lambda)^* D(\{d_2^{n-1}a\}, \mathrm{id})(\lambda^* f)(d_0 a) + \sum_{i=1}^{n-1}(-1)^i(\lambda^* f)(d_i a)\\ &+ (-1)^n(\tau_1\lambda)^* D(\mathrm{id}, \{d_0^{n-1}a\})(\lambda^* f)(d_n a) = \partial(\lambda^* f)(a).\end{aligned}$$

□

Lemma 3.1.8. A morphism $(\mathrm{id}, \mu)\colon (X, D) \to (X, D')$ induces a homomorphism:

$$F^n(\mathrm{id}, \mu) := \mu_*\colon F^n(X, D) \longrightarrow F^n(X, D'),$$

defined for $f \in F^n(X, D)$ and $a \in X_n$ by postcomposition with μ:

$$\mu_* f(a) = \mu(f(a)).$$

Proof. We have to check that $\mu_* f \in F^n(X, D')$, i.e. $\mu_* f(a) \in D'(\{d_1^{n-1}a\})$. We have $f(a) \in D(\{d_1^{n-1}a\})$ and $\mu\colon D \to D'$, so:

$$\mu_* f(a) = \mu(f(a)) \in D'(\{d_1^{n-1}a\}).$$

Now, we have to prove the compatibility with the coboundary, i.e. $\partial\mu_* = \mu_*\partial$. Given $f \in F^{n-1}(X, D)$ and $a \in X_n$ and using the fact that λ is a simplicial map, we have:

$$\begin{aligned}\partial(\mu_* f)(a) =&D'(\{d_2^{n-1}a\}, \mathrm{id})(\mu_* f)(d_0 a) + \sum_{i=1}^{n-1}(-1)^i(\mu_* f)(d_i a) + (-1)^n D'(\mathrm{id}, \{d_0^{n-1}a\})(\mu_* f)(d_n a) =\\ =&D'(\{d_2^{n-1}a\}, \mathrm{id})\mu(f(d_0 a)) + \sum_{i=1}^{n-1}(-1)^i\mu(f(d_i a)) + (-1)^n D'(\mathrm{id}, \{d_0^{n-1}a\})\mu(f(d_n a)).\end{aligned}$$

From there we can use the fact that $\mu\colon D \to D'$ is a natural transformation, which implies $D'(-,-)\mu = \mu D(-,-)$, giving us:

$$\begin{aligned}\partial(\mu_* f)(a) =&\mu D(\{d_2^{n-1}a\}, \mathrm{id})(f(d_0 a)) + \sum_{i=1}^{n-1}(-1)^i\mu(f(d_i a)) + (-1)^n\mu D(\mathrm{id}, \{d_0^{n-1}a\})(f(d_n a)) =\\ =&\mu\left(D(\{d_2^{n-1}a\}, \mathrm{id})f(d_0 a) + \sum_{i=1}^{n-1}(-1)^i f(d_i a) + (-1)^n D(\mathrm{id}, \{d_0^{n-1}a\})f(d_n a)\right) =\\ =&\mu(\partial f(a)) = \mu_*\partial f(a).\end{aligned}$$

□

Proposition 3.1.9. The cochains define a functor $F^n\colon \mathbf{Nat} \to \mathbf{Ab}$. Any morphism $(\lambda, \mu)\colon (X, D) \to (X', D')$ induces the following homomorphism on cochains:

$$F^n(\lambda, \mu) := \mu_*\lambda^*\colon F^n(X, D) \longrightarrow F^n(X', D').$$

Proof. It is straightforward to see that $F^n(\mathrm{id},\mathrm{id}) = \mathrm{id}_{F^n(X,D)}$. In the case of compositions, given $(\lambda,\mu)\colon (X,D) \to (X',D')$ and $(\lambda',\mu')\colon (X',D') \to (X'',D'')$ we have:

$$F^n((\lambda',\mu')(\lambda,\mu)) = F^n(\lambda\lambda', \mu' \circ (\tau_1\lambda')^*\mu) = \mu'_*((\tau_1\lambda')^*\mu)_*\lambda'^*\lambda^*.$$

But we can write $((\tau_1\lambda')^*\mu)_*\lambda'^* = \lambda'^*\mu_*$, because for $f \in F^n(X', (\tau_1\lambda)^*D)$ and $a \in X'_n$:

$$\lambda'^*\mu_*f(a) = ((\tau_1\lambda')^*\mu)_*f(\lambda'(a)) = ((\tau_1\lambda')^*\mu)_*\lambda'^*f(a),$$

and that gives us:

$$F^n((\lambda',\mu')(\lambda,\mu)) = \mu'_*\lambda'^*\mu_*\lambda^* = F^n(\lambda',\mu')F^n(\lambda,\mu).$$

□

Having studied these basic properties of our cochains, we can now define our cohomology:

Definition 3.1.10. The cohomology of a simplicial set X with coefficients on a natural system D on τ_1X, $H^n(X,D)$, is the cohomology of the cochain complex $F^n(X,D)$ with the coboundary ∂. Therefore, the cohomology defines functors $H^n\colon \mathbf{Nat} \to \mathbf{Ab}$.

3.2 Simplicial properties of the cohomology

We explore further properties of our cohomology by studying its relation with a generalization of Eilenberg-MacLane spaces. The theorems in this section follow closely [4, §17], with some important changes to adapt them to our situation.

Definition 3.2.1. Fixing $n \geq 0$ we define the simplicial set $L_\bullet(D,n)$ with:

$$L_q(D,n) = \{(a,b) \mid a \in \mathrm{N}_q\tau_1X,\ b \in F^n([q], a^*D)\}\,.$$

In the second component, we are considering $[q]$ as a category and taking $a \in \mathrm{N}_q\tau_1X$ as a functor $a\colon [q] \to \tau_1X$, so a^*D makes sense as a natural system on $[q]$.

We define the face and degeneracy maps of this simplicial set as:

$$\begin{array}{rccl} d_i\colon & L_q(D,n) & \longrightarrow & L_{q-1}(D,n), \\ & (a,b) & \longmapsto & d_i(a,b) = (d_ia, (d^i)^*b), \end{array}$$

$$\begin{array}{rccl} s_i\colon & L_{q-1}(D,n) & \longrightarrow & L_q(D,n), \\ & (a,b) & \longmapsto & s_i(a,b) = (s_ia, (s^i)^*b). \end{array}$$

With $d^i\colon [q-1] \to [q]$ and $s^i\colon [q] \to [q-1]$ as defined in (2.2.2), but thinking of them as functors. In the case of the coface map, it induces $d^i\colon \mathrm{N}_n([q-1]) \to \mathrm{N}_n([q])$, which in turn by Lemma 3.1.7 induces:

$$(d^i)^*\colon F^n([q], a^*D) \longrightarrow F^n([q-1], (d^i)^*a^*D).$$

This face map is well-defined because we know that $(d^i)^*a^*D = (ad^i)^*D = (d_ia)^*D$ by the definition of face map of the nerve, which makes $F^n([q-1], (d^i)^*a^*D) = F^n([q-1], (d_ia)^*D)$ as desired. The degeneracies work analogously.

Considering the simplicial set as a functor, $L(D,n)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$, any simplicial map $\alpha\colon [q] \to [m]$ induces:

$$\begin{array}{rccl} L(D,n)(\alpha)\colon & L_m(D,n) & \longrightarrow & L_q(D,n), \\ & (a,b) & \longmapsto & L(D,n)(\alpha)\,(a,b) = (N\tau_1X(\alpha)\,a,\ \alpha^*b). \end{array} \tag{3.2.2}$$

This simplicial set comes equipped with a projection map onto the first component:

$$\begin{aligned} p\colon \quad L(D,n) &\longrightarrow \mathrm{N}\tau_1 X, \\ (a,b) &\longmapsto a. \end{aligned}$$

Definition 3.2.3. We want to consider the set of simplicial maps between X and $L(D,n)$ which make the following triangle commute:

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\Psi"'] & & L(D,n), \arrow[dl, "p"] \\ & \mathrm{N}\tau_1 X & \end{tikzcd}$$

with Ψ as defined in (2.3.8) and p being the projection onto the first component.

This set will be denoted by $\hom_{\mathrm{N}\tau_1 X}(X, L(D,n))$ and any element g can be described as:

$$\begin{aligned} g\colon \quad X_q &\longrightarrow L_q(D,n), \\ a &\longmapsto g(a) = (\Psi(a), g(a)_2). \end{aligned}$$

This set carries an abelian group structure defined as follows, for $g, h \in \hom_{\mathrm{N}\tau_1 X}(X, L(D,n))$:

$$(g+h)(a) = (\Psi(a), g(a)_2 + h(a)_2).$$

Having defined this abelian group, our next goal is to find an isomorphism:

$$\begin{tikzcd} \hom_{\mathrm{N}\tau_1 X}(X, L(D,n)) \arrow[r, shift left, "\phi"] & F^n(X,D). \arrow[l, shift left, "\varphi"] \end{tikzcd}$$

Lemma 3.2.4. We define the morphism $\phi\colon \hom_{\mathrm{N}\tau_1 X}(X, L(D,n)) \to F^n(X,D)$ as follows, given $g \in \hom_{\mathrm{N}\tau_1 X}(X, L(D,n))$ and $a \in X_n$:

$$\phi(g)(a) = g(a)_2(\mathrm{id}_{[n]}),$$

where $\mathrm{id}_{[n]} \in \mathrm{N}_n([n])$ and $(g(a))_2 \in F^n([n], (\Psi(a))^* D)$.

Proof. We should check if it is correctly defined. That means proving $\phi(g)$ is an n-cochain, i.e. it satisfies $\phi(g)(a) \in D(\{d_1^{n-1} a\})$:

$$\phi(g)(a) = g(a)_2(\mathrm{id}_{[n]}) \in (\Psi(a))^* D(d_1^{n-1} \mathrm{id}_{[n]}).$$

This natural system makes sense as $\Psi(a) \in N_n \tau_1 X$, thus $\Psi(a)\colon [n] \to \tau_1 X$. This means $(\Psi(a))^* D(d_1^{n-1} \mathrm{id}_{[n]}) = D(\Psi(a d_1^{n-1} \mathrm{id}_{[n]}))$ and from this point we can use the fact that a is a simplicial map and we get:

$$D(\Psi(a d_1^{n-1} \mathrm{id}_{[n]})) = D(\Psi(d^{n-1} a(\mathrm{id}_{[n]}))) = D(\Psi(d_1^{n-1} a)).$$

Then, considering Ψ as a functor and using the fact that $d_1^{n-1} a \in X_1$, we have:

$$D(\Psi(d_1^{n-1} a)) = D(\{d_1^{n-1} a\})$$

as desired. $\square$

Lemma 3.2.5. We define the morphism $\varphi\colon F^n(X,D) \to \hom_{\mathrm{N}\tau_1 X}(X, L(D,n))$, for $f \in F^n(X,D)$ and $a \in X_q$, as:

$$\varphi(f)(a) = (\Psi(a), a^* f),$$

where $a\colon \mathrm{N}([q]) \to X$ induces $a^*\colon F^n(X,D) \to F^n([q], (\tau_1(a))^* D)$.

Proof. First, we need to prove that $F^n([q], (\tau_1 a)^* D)$ is equal to $F^n([q], (\Psi(a))^* D)$. Given $v \in \mathrm{N}_n[q]$, we have:

$$a^* f(v) = f(a(v)) \in D(\{d_1^{n-1} a(v)\}) = D(\{a(d_1^{n-1} v)\}),$$

and, as $a(d_1^{n-1} v) \in X_1$, we can use the fact that Ψ restricted to X_1, $\Psi\colon X_1 \to \mathrm{N}_1\tau_1 X = \mathrm{Mor}(\tau_1 X)$, sends each 1-simplex to its equivalence class, the same that τ_1 does.

$$(\Psi(a)) * D(\{d_1^{n-1} v\}) = D(\{a(d_1^{n-1} v)\}) = (\tau_1 a)^* D(\{d_1^{n-1} v\}).$$

We also need to check that $\varphi(f)$ is a simplicial map. We will only check it for faces, as degeneracies are analogous:

$$\begin{aligned} d_i(\varphi(f)(a)) &= d_i(\Psi(a), a^* f) = (d_i \Psi(a), (d^i)^* a^* f) = \\ &= (\Psi(d_i a), (a d_i)^* f) = (\Psi(d_i a), (d_i a)^* f) = \varphi(f)(d_i a). \end{aligned}$$

□

With these morphisms defined and verified, we now turn to proving that they constitute an isomorphism.

Theorem 3.2.6. φ and ϕ are isomorphisms and inverses to each other.

Proof. We will prove that both $\phi\varphi$ and $\varphi\phi$ are identities.

First, for $f \in F^n(X,D)$ and $a \in X_n$:

$$(\phi\varphi)(f)(a) = \phi(\varphi(f))(a) = \varphi(f)(a)_2(\mathrm{id}_{[n]}) = a^* f(\mathrm{id}_{[n]}) = f(a(\mathrm{id}_{[n]})) = f(a).$$

Conversely, for $g \in \hom_{\mathrm{N}\tau_1 X}(X, L(D,n))$ and $a \in X_q$:

$$(\varphi\phi)(g)(a) = \varphi(\phi(g))(a) = (\Psi(a), a^*(\phi(g))).$$

To prove that $\varphi\phi = \mathrm{id}$, the only thing left to be checked is that $a^*(\phi(g)) = g(a)_2$, so we will study only the second component.

Taking $\alpha \in \mathrm{N}_n([q])$, we can consider $\alpha\colon [n] \to [q]$. With that, and knowing that both a and g are simplicial maps, we get:

$$\begin{aligned} a^*(\phi(g))(\alpha) &= \phi(g)(a(\alpha)) = g(a(\alpha))_2(\mathrm{id}_{[n]}) = g(X(\alpha)a)_2(\mathrm{id}_{[n]}) = \\ &= (L(D,n)(\alpha)\, g(a))_2(\mathrm{id}_{[n]}) = \alpha^*(g(a))_2(\mathrm{id}_{[n]}) = g(a)_2(\alpha\, \mathrm{id}_{[n]}) = g(a)_2(\alpha), \end{aligned}$$

where we are considering the simplicial sets X and $L(D,n)$ as functors and using the definition of $L(D,n)$ on morphisms as in (3.2.2). □

Now, we want to restrict this isomorphism to cocycles:

Definition 3.2.7. Fixing $n \geq 0$ we define the simplicial set $K_\bullet(D,n)$ as the following subsimplicial set of $L(D,n)$:

$$K_q(D,n) = \{(a,b) \mid a \in \mathrm{N}_q\tau_1 X,\ b \in Z^n([q], a^*D)\},$$

which is correctly defined because for $(a,b) \in K_q(D,n)$, $(d^i)^*b$ and $(s^i)^*b$ are also cocycles due to the functoriality of $F^n(-,-)$ with respect to ∂.

Theorem 3.2.8. The restrictions of φ and ϕ define an isomorphism:

$$\begin{tikzcd} \hom_{\mathrm{N}\tau_1 X}(X, K(D,n)) \arrow[r, shift left, "\phi"] & Z^n(X,D). \arrow[l, shift left, "\varphi"] \end{tikzcd}$$

Proof. Taking $f \in Z^n(X,D)$ we want to check that for any $a \in X_q$, $\varphi(f)(a) \in K_q(D,n)$, which means we need to study the second component to see if it is a cocycle:

$$\partial(\varphi(f)(a))_2 = \partial a^*(f) = a^*(\partial(f)) = a^*(0) = 0,$$

where we have used the definition of φ, the functoriality of a^* and the fact that f is a cocycle.

Conversely, given $\varphi(f) \in \hom_{\mathrm{N}\tau_1 X}(X, K(D,n))$, we want to check $f \in Z^n(X,D)$. Taking $a \in X_{n+1}$, we can think of it as a map $a\colon \mathrm{N}_{n+1} \to X$ with $a(\mathrm{id}_{[n+1]}$, so we have:

$$\partial f(a) = \partial f(a(\mathrm{id}_{[n+1]})) = \partial(a^*f)(\mathrm{id}_{[n+1]}) = \partial(\varphi(f)(a))_2(\mathrm{id}_{[n+1]}) = 0.$$

□

3.3 Properties of $L(D,n)$

We prove that both $L(D,n)$ and $K(D,n)$ are quasicategories. This result could be used to prove the invariance by quasiequivalences of the cohomology.

The first result below is from [3, Lemma 0032] and the second one is adapted from [5, Section 08NT].

Lemma 3.3.1. The nerve of a category, $\mathrm{N}\mathbf{C}$, is a quasicategory. Moreover, each inner horn admits a unique filler morphism:

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d, hook] & \mathrm{N}\mathbf{C} \\ \Delta[n] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd} \quad \text{for } 0 < i < n.$$

Theorem 3.3.2. $L(D,n)$ is a quasicategory.

Proof. For some $0 < k < q$, a map σ from $\Lambda^k[q]$ is defined by the images of the $(q-1)$-simplices $d_i\,\mathrm{id}_{[q]}$ for $i = 0, \dots, \widehat{k}, \dots q$. We will denote their images by:

$$\begin{array}{rccl} \sigma\colon & \Lambda^k[q] & \longrightarrow & L(D,n), \\ & d_i\,\mathrm{id}_{[q]} & \longmapsto & (u_i, v_i) \quad \in L_{q-1}(D,n), \end{array}$$

which satisfy the relations of the simplices of the horn, namely $d_i(u_j, v_j) = d_i\sigma(d_j\,\mathrm{id}_{[q]}) = d_{j-1}\sigma(d_i\,\mathrm{id}_{[q]}) = d_{j-1}(u_i, v_i)$ for $i < j$ and $i, j \neq k$.

To prove this theorem we need to find a q-simplex $(u,v) \in L_q(D,n)$ such that $d_i(u,v) = (u_i, v_i)$ for each $i = 0, \ldots, \widehat{k}, \ldots q$.

The component $u \in \mathrm{N}_q\tau_1 X$ of (u,v) can be defined by Lemma 3.3.1 and satisfies $d_i u = u_i$. So the only thing missing is finding a cochain $v \in F^n([q], u^*D)$ such that $(d^i)^* v = v_i$ for all $i \neq k$. We will construct this cochain in two steps.

The first step is to define $c \in F^n([q], u^*D)$ satisfying $(d^i)^* c = v_i$ for all $i < k$. We will do so by induction on r, $0 \leq r < k$, to define a $c^r \in F^n([q], u^*D)$ such that $(d^i)^* c^r = v_i$ for $i \leq r$.

To define the initial case, $c^0 \in F^n([q], u^*D)$, we are going to study the cochain $(s^0)^* v_0 \in F^n([q], (s_0 u_0)^* D) = F^n([q], (s_0 d_0 u)^* D)$. Taking $\tau \in \mathrm{N}_n[q]$, we have:

$$\begin{aligned} c^0(\tau) \in& u^* D(d_1^{n-1}\tau) = D(\{u d_1^{n-1}\tau\}) = D(\{u\tau(d^1)^{n-1}\}), \\ (s^0)^* v_0(\tau) \in& (s_0 d_0 u)^* D(d_1^{n-1}\tau) = D\{s_0 d_0 u d_1^{n-1}\tau\}) = D(\{u d^0 s^0 \tau (d^1)^{n-1}\}). \end{aligned}$$

We want to study when those two abelian groups are the same, for that we need to study and compare $\tau(d^1)^{n-1}$ and $d^0 s^0 \tau (d^1)^{n-1}$. As $\tau \in \mathrm{N}_n[q]$, it can be viewed as morphism $\tau\colon [n] \to [q]$, thus:

$$[1] \xrightarrow{(d^1)^{n-1}} [n] \xrightarrow{\tau} [q],$$

$$[1] \xrightarrow{(d^1)^{n-1}} [n] \xrightarrow{\tau} [q] \xrightarrow{s^0} [q-1] \xrightarrow{d^0} [q].$$

The category [1] is the category with two objects and one morphism between them, so in both cases we are mapping that morphism, $1 \geq 0$, to a morphism in $[q]$. Taking into account the definitions of the cofaces and codegeneracies we have $(d^1)^{n-1}(0) = 0$, $(d^1)^{n-1}(1) = n$, $d^0 s^0(0) = 1$ and $d^0 s^0(l) = l$ for any $l > 0$. This means that the source (image of 1) and target (image of 0) of the morphisms in $[q]$ are:

$$\begin{aligned} \tau(d^1)^{n-1}(1) =& \tau(n), \\ \tau(d^1)^{n-1}(0) =& \tau(0), \\ d^0 s^0 \tau (d^1)^{n-1}(1) =& \begin{cases} 1 & \text{if} \quad \tau(n) = 0, \\ \tau(n) & \text{if} \quad \tau(n) > 0, \end{cases} \\ d^0 s^0 \tau (d^1)^{n-1}(0) =& \begin{cases} 1 & \text{if} \quad \tau(0) = 0, \\ \tau(0) & \text{if} \quad \tau(0) > 0. \end{cases} \end{aligned}$$

Thus, the images of the morphism $1 \geq 0$ are $u(d^1)^{n-1}(1 \geq 0) = \tau(n) \geq \tau(0)$ and $d^0 s^0 \tau (d^1)^{n-1}(1 \geq 0) = \max\{\tau(n), 1\} \geq \max\{\tau(0), 1\}$.

With all of this explained we can define $c^0 \in F^n([q], u^*D)$ with $\tau \in \mathrm{N}_n[q]$:

$$c^0(\tau) = \begin{cases} 0 & \text{if} \quad \tau(n) = 0, \\ (s^0)^* v_0(\tau) & \text{if} \quad \tau(n) > 0, \tau(0) > 0, \\ D(u(1 \geq 0), \mathrm{id})((s^0)^* v_0(\tau)) & \text{if} \quad \tau(n) > 0, \tau(0) = 0. \end{cases}$$

In any of the three cases, c^0 satisfies the cochain condition. The first case is obvious. For the second one we need to use the previous explanation:

$$(s^0)^*v_0(\tau) \in D(\{ud^0s^0\tau(d^1)^{n-1}\}) = D(\{u(\tau(n) \geq \tau(0))\}) = D(\{u\tau(d^1)^{n-1}\}).$$

Finally, in the third one we have:

$$(s^0)^*v_0(\tau) \in D(\{ud^0s^0\tau(d^1)^{n-1}\}) = D(\{u(\tau(n) \geq 1)\}),$$

then applying the map $D(u(1 \geq 0), \mathrm{id})\colon D(\{u(\tau(n) \geq 1)\}) \to D(\{u(1 \geq 0) \circ u(\tau(n) \geq 1)\}) = D(\{u(\tau(n) \geq 0)\})$ we get what we wanted.

Now, we want to prove that $(d^0)^*c^0 = v_0$. Taking $\tau \in \mathrm{N}_n[q-1]$:

$$(d^0)^*c^0(\tau) = c^0(d^0\tau),$$

and using the definition of $d^0(j) = j+1$ for all $0 \leq j \leq q-1$, we know that $(d^0\tau)(0) > 0$, which means we are in the second case of the definition of c^0, so:

$$c^0(d^0\tau) = ((s^0)^*v_0)(d^0\tau) = v_0(s^0d^0\tau) = v_0(\tau).$$

Now, for the inductive step, we suppose there exists a c^r as desired for each $r < k-1$, and we define a cochain $y^r \in F^n([q], u^*D)$ such that $c^{r+1} = c^r + y^r$ satisfies $(d^i)^*(c^r + y^r) = v_i$ for all $i \leq r+1$.

Analogously to the construction of c^0, before we define y^r we need to study the cochain:

$$(y')^r = (s^{r+1})^*(v_{r+1} - (d^{r+1})^*c^r).$$

By hypothesis $c^r \in F^n([q], u^*D)$, thus $(d^{r+1})^*c^r \in F^n([q-1], (d_{r+1}u)^*D) = F^n([q-1], u^*_{r+1}D)$, which means $(y')^r \in F^n([q], (s_{r+1}d_{r+1}u)^*D)$.

Taking $\tau \in \mathrm{N}_n[q]$, we compare y^r and $(y')^r$:

$$\begin{aligned} y^r(\tau) \in& u^*D(d_1^{n-1}\tau) = D(\{ud_1^{n-1}\tau\}) = D(\{u(\tau(n) \geq \tau(0))\}),\\ (y')^r(\tau) \in& (s_{r+1}d_{r+1}u)^*D(d_1^{n-1}\tau) = D\{s_{r+1}d_{r+1}ud_1^{n-1}\tau\}) = D(\{ud^{r+1}s^{r+1}(\tau(n) \geq \tau(0))\}). \end{aligned}$$

Using the definition of the cochains and codegeneracies we know:

$$d^{r+1}s^{r+1}(j) = \begin{cases} j & \text{if } \ j < r+1,\\ r+2 & \text{if } \ j = r+1,\\ j & \text{if } \ j > r+1, \end{cases}$$

which means:

$$d^{r+1}s^{r+1}(\tau(n) \geq \tau(0)) = \begin{cases} \mathrm{id}_{r+1} & \text{if } \ \tau(n) = \tau(0) = r+1,\\ \tau(n) \geq \tau(0) & \text{if } \ \tau(n) \neq r+1, \tau(0) \neq r+1,\\ r+2 \geq \tau(0) & \text{if } \ \tau(n) = r+1, \tau(0) < r+1,\\ \tau(n) \geq r+2 & \text{if } \ \tau(n) > r+1, \tau(0) = r+1. \end{cases}$$

Now, we can define $y^r \in F^n([q], u^*D)$ for any $\tau \in \mathrm{N}_n[q]$:

$$y^r(\tau) = \begin{cases} 0 & \text{if } \ \tau(n) = r+1,\\ (y')^r & \text{if } \ \tau(n) \neq r+1, \tau(0) \neq r+1,\\ D(u(r+2 \geq r+1), \mathrm{id})(y')^r & \text{if } \ \tau(n) > r+1, \tau(0) = r+1. \end{cases}$$

This cochain is compatible with the natural system. In the first two cases it is straightforward, and in the third one we have the morphism:

$$D(u(r+2 \geq r+1), \mathrm{id})\colon D(\{u(\tau(n) \geq r+2)\}) \longrightarrow D(\{u(r+2 \geq r+1) \circ u(\tau(n) \geq r+2)\}),$$

with $u(r+2 \geq r+1) \circ u(\tau(n) \geq r+2) = u(\tau(n) \geq r+1) = u(\tau(n) \geq \tau(0)$.

Before defining c^{r+1}, we also want to check the faces of y^r. Taking $\tau \in \mathrm{N}_n[q-1]$ and for $i \leq r$ we have $(d^i)^* y^r(\tau) = y^r(d^i\tau)$, so we will need to distinguish between the distinct cases of $d^i\tau$:

- *Case 1:* if $(d^i\tau)(n) = r+1$:
$$y^r(d^i\tau) = 0.$$
- *Case 2:* if $(d^i\tau)(n) \neq r+1$ and $(d^i\tau)(0) \neq r+1$:
$$\begin{aligned} y^r(d^i\tau) =& (y')^r(d^i\tau) = (s^{r+1})^*(v_{r+1} - (d^{r+1})^* c^r)(d^i\tau) = \\ =& (d^i)^*(s^{r+1})^*(v_{r+1} - (d^{r+1})^* c^r)(\tau) = (s^r)^*(d^i)^*(v_{r+1} - (d^{r+1})^* c^r)(\tau) = \\ =& (s^r)^*((d^i)^* v_{r+1} - (d^i)^*(d^{r+1})^* c^r) = (s^r)^*((d^i)^* v_{r+1} - (d^r)^*(d^i)^* c^r)(\tau) = \\ =& (s^r)^*((d^i)^* v_{r+1} - (d^r)^* v_i)(\tau) = 0, \end{aligned}$$
in the last equality we have used the fact that $d_i(u_j, v_j) = d_{j-1}(u_i, v_i)$ for $i < j$.
- *Case 3:* if $(d^i\tau)(n) > r+1$ and $(d^i\tau)(0) = r+1$:
$$\begin{aligned} y^r(d^i\tau) =& ((d^i)^* D)(u(r+2 \geq r+1), \mathrm{id})(y')^r(d^i\tau) = (d^i)^* \left(D(u(r+2 \geq r+1), \mathrm{id})(y')^r\right)(\tau) = \\ =& \left((d^i)^* D)(u(r+2 \geq r+1), \mathrm{id})\right)(d^i)^*(y')^r(\tau) = 0. \end{aligned}$$
In this calculation we have used that the natural system of $(d^i)^* y^r \in F^n([q-1], (d_i u)^* D)$ is precisely $(d^i)^* u^* D$ and in the last equality we have applied case 2, which proved $(d^i)^*(y')^r = 0$.

The three cases give us $(d^i)^* y^r = 0$ for $i \leq r$.

Given $\tau \in \mathrm{N}_n[q-1]$, the case $i = r+1$ is $(d^{r+1})^* y^r(\tau) = y^r(d^{r+1}\tau)$ and we know $d^{r+1}(j) \neq r+1$ for all $0 \leq j \leq q-1$, which means we are in the second case of y^r:

$$y^r(d^{r+1}\tau) = (y')^r(d^{r+1}\tau) = (d^{r+1})^*(s^{r+1})^*(v_{r+1} - (d^{r+1})^* c^r)(\tau) = (v_{r+1} - (d^{r+1})^* c^r)(\tau).$$

We use y^r to define $c^{r+1} = c^r + y^r$, satisfying the condition we asked for:

$$\begin{aligned} (d^i)^* c^{r+1} &= (d^i)^* c^r + (d^i)^* y^r = v_i + 0 = v_i, \text{ for } i \leq r, \\ (d^{r+1})^* c^{r+1} &= (d^{r+1})^* c^r + (d^{r+1})^* y^r = (d^{r+1})^* c^r + v_{r+1} - (d^{r+1})^* c^r = v_{r+1}. \end{aligned}$$

We can define $c := c^{k-1}$ and the first step is completed.

The second step is again by induction on r, $0 \leq r \leq n-k$, to define $b^r \in F^n([q], u^* D)$ such that $(d^i)^* b^r = v_i$ for $i < k$ and $i > n-r$.

For $r = 0$, we define $b^0 = c$, which clearly satisfies the condition. Then, assuming the case $r \leq n - k - 1$, we need to define $x^r \in F^n([q], u^*D)$ such that we can define $c^{r+1} = c^r + x^r$ satisfying the desired condition. However, before defining x^r we need to study another cochain:

$$(x')^r = (s^{n-r-1})^*(v_{n-r} - (d^{n-r})^* b^r) \in F^n([q], (s_{n-r-1}d_{n-r}u)^*D).$$

Taking $\tau \in \mathrm{N}_n[q]$ and comparing x^r and $(x')^r$:

$$\begin{aligned} x^r(\tau) \in& u^*D(d_1^{n-1}\tau) = D(\{ud_1^{n-1}\tau\}) = D(\{u(\tau(n) \geq \tau(0))\}), \\ (x')^r(\tau) \in& (s_{n-r-1}d_{n-r}u)^*D(d_1^{n-1}\tau) = D\{s_{n-r-1}d_{n-r}ud_1^{n-1}\tau\}) = D(\{ud^{n-r}s^{n-r-1}(\tau(n) \geq \tau(0))\}). \end{aligned}$$

Developing the last morphism:

$$d^{n-r}s^{n-r-1}(j) = \begin{cases} j & \text{if} \quad j < n - r, \\ n - r - 1 & \text{if} \quad j = n - r, \\ j & \text{if} \quad j > n - r, \end{cases}$$

$$d^{n-r}s^{n-r-1}(\tau(n) \geq \tau(0)) = \begin{cases} \mathrm{id}_{n-r-1} & \text{if} \quad \tau(n) = \tau(0) = n - r, \\ \tau(n) \geq \tau(0) & \text{if} \quad \tau(n) \neq n - r, \tau(0) \neq n - r, \\ n - r - 1 \geq \tau(0) & \text{if} \quad \tau(n) = n - r, \tau(0) < n - r, \\ \tau(n) \geq n - r - 1 & \text{if} \quad \tau(n) > n - r, \tau(0) = n - r. \end{cases}$$

With this we can define $x^r \in F^n([q], u^*D)$ for any $\tau \in \mathrm{N}_n[q]$:

$$x^r(\tau) = \begin{cases} 0 & \text{if} \quad \tau(0) = n - r, \\ (x')^r & \text{if} \quad \tau(n) \neq n - r, \tau(0) \neq n - r, \\ D(\mathrm{id}, u(n - r \geq n - r - 1))(x')^r & \text{if} \quad \tau(n) = n - r, \tau(0) < n - r, \end{cases}$$

which satisfies $x^r \in F^n([q], u^*D)$ for all three cases. The first two are straightforward and the third one is a consequence of the morphism:

$$D(\mathrm{id}, u(n-r \geq n-r-1)) \colon D(\{u(n-r-1 \geq \tau(0))\}) \longrightarrow D(\{u(n-r-1 \geq \tau(0)) \circ u(n-r \geq n-r-1)\}).$$

Now we are going to study the faces of x^r. Given $\tau \in \mathrm{N}_n[q-1]$, we begin by studying the faces for $i < k$:

- *Case 1:* if $(d^i\tau)(n) = n - r$:
$$x^r(d^i\tau) = 0.$$
- *Case 2:* if $(d^i\tau)(n) \neq n - r$ and $(d^i\tau)(0) \neq n - r$:
$$\begin{aligned} x^r(d^i\tau) =& (x')^r(d^i\tau) = (d^i)^*(s^{n-r-1})^*(v_{n-r} - (d^{n-r})^*b^r)(\tau) = \\ =& (s^{n-r-2})^*((d^i)^*v_{n-r} - (d^i)^*(d^{n-r})^*b^r)(\tau) = \\ =& (s^{n-r-2})^*((d^i)^*v_{n-r} - (d^{n-r-1})^*(d^i)^*b^r)(\tau) = \\ =& (s^{n-r-2})^*((d^i)^*v_{n-r} - (d^{n-r-1})^*v_i)(\tau) = 0, \end{aligned}$$
in the last equality we have used the fact that $d_i(u_j, v_j) = d_{j-1}(u_i, v_i)$ for $i < j$.
- *Case 3:* if $(d^i\tau)(n) = n - r$ and $(d^i\tau)(0) < r + 1$:
$$\begin{aligned} x^r(d^i\tau) =& ((d^i)^*D)(\mathrm{id}, u(n - r \geq n - r - 1))(x')^r(d^i\tau) = (d^i)^*\left(D(\mathrm{id}, u(n - r \geq n - r - 1))(x')^r\right)(\tau) = \\ =& \left(((d^i)^*D)(\mathrm{id}, u(n - r \geq n - r - 1))\right)(d^i)^*(x')^r(\tau) = 0. \end{aligned}$$
In this calculation we have used that the natural system of $(d^i)^*x^r \in F^n([q-1], (d_iu)^*D)$ is precisely $(d^i)^*u^*D$ and in the last equality we have applied case 2, which proved $(d^i)^*(x')^r = 0$.

The faces for $i > n - r$ are:

- *Case 1:* if $(d^i\tau)(n) = n - r$:
$$x^r(d^i\tau) = 0.$$
- *Case 2:* if $(d^i\tau)(n) \neq n - r$ and $(d^i\tau)(0) \neq n - r$:
$$\begin{aligned} x^r(d^i\tau) =&(x')^r(d^i\tau) = (d^i)^*(s^{n-r-1})^*(v_{n-r} - (d^{n-r})^*b^r)(\tau) = \\ =&(s^{n-r-1})^*((d^{i-1})^*v_{n-r} - (d^{i-1})^*(d^{n-r})^*b^r)(\tau) = \\ =&(s^{n-r-1})^*((d^{i-1})^*v_{n-r} - (d^{n-r})^*(d^i)^*b^r)(\tau) = \\ =&(s^{n-r-1})^*((d^{i-1})^*v_{n-r} - (d^{n-r})^*v_i)(\tau) = 0, \end{aligned}$$
in the last equality we have used the fact that $d_i(u_j, v_j) = d_{j-1}(u_i, v_i)$ for $i < j$.
- *Case 3:* if $(d^i\tau)(n) = n - r$ and $(d^i\tau)(0) < r + 1$:
$$x^r(d^i\tau) = 0.$$
This case is analogous to the previous case 3.

The last face we need to check is $i = n-r$, in which case $d^{n-r}(j) \neq n-r$ for all $0 \leq j \leq q-1$, so we are in the second case of x^r:

$$x^r(d^{n-r}\tau) = (x')^r(d^{n-r}\tau) = (d^{n-r})^*(s^{n-r-1})^*(v_{n-r} - (d^{n-r})^*b^r)(\tau) = (v_{n-r} - (d^{n-r})^*b^r)(\tau).$$

Summarizing, the faces of x^r are:

$$\begin{aligned} (d^i)^*x^r =&0, \text{ for } i < k \text{ and } i > n - r, \\ (d^{n-r})^*x^r =&v_{n-r} - (d^{n-r})^*b^r. \end{aligned}$$

With x^r we define $b^{r+1} = b^r + x^r$, satisfying:

$$\begin{aligned} (d^i)^*b^{r+1} &= (d^i)^*b^r + (d^i)^*x^r = v_i + 0 = v_i, \text{ for } i < k \text{ and } i > n - r, \\ (d^{n-r})^*b^{r+1} &= (d^{n-r})^*b^r + (d^{n-r})^*x^r = (d^{n-r})^*b^r + v_{n-r} - (d^{n-r})^*b^r = v_{n-r}. \end{aligned}$$

Finally, we can define $v \coloneqq b^{n-k}$ and we have v as desired:

$$(u, v) \in K_q(D, n) \text{ with } d_i(u, v) = (u_i, v_i) \text{ for } i \neq k.$$

□

This proof could have been identically done with $K(D, n)$ instead of $L(D, n)$, as all the facts we have used about $L(D, n)$ are shared by $K(D, n)$.

Theorem 3.3.3. *$K(D, n)$ is a quasicategory.*

3.4 Future directions

Further investigation about the theme of this memoir should probably be aimed at proving the invariance by quasiequivalences of the cohomology defined here. This result is an almost straightforward consequence of two theorems, one of them is Theorem 3.3.3, but we have not been able to prove the other one. Nevertheless, in this section we present both the invariance theorem and the needed previous result.

Defining $p' : X \times \Delta[1] \to X$ to be the projection $p(x, y) = x$ for all $x \in X_q$ and $y \in \Delta[1]_q$ for any q, we have:

Definition 3.4.1. Two maps $f, g \in \hom_{\mathrm{N}\tau_1 X}(X, K(D, n))$ are said to be homotopic if there exists $H\colon X \times \Delta[1] \to K(D, n)$ satisfying for any $x \in X_q$:

$$\begin{aligned} H(x, s_0^q(0)) =& f(x), \\ H(x, s_0^q(1)) =& g(x). \end{aligned}$$

And making the following square commute:

$$\begin{tikzcd} X \times \Delta[1] \arrow[r, "p'"] \arrow[d, "H"'] & X \arrow[d, "\Psi"] \\ K(D,n) \arrow[r, "p"'] & \mathrm{N}\tau_1 X. \end{tikzcd}$$

Theorem 3.4.2. Given $f, g \in \hom_{\mathrm{N}\tau_1 X}(X, K(D, n))$, they are homotopic if and only if $\phi(f)$ and $\phi(g)$ are cohomologous.

This result is an extension to natural systems of the one present in [4, §17].

After this previous result we can give a sketch of the proof for the invariance theorem:

Theorem 3.4.3. Let $f\colon X \to Y$ be a quasiequivalence and D a natural system in $\tau_1 Y$. Then $f^n\colon H^n(Y, D) \to H^n(X, f^* D)$ is an isomorphism for all $n \in \mathbb{Z}$.

Proof. The category of simplicial sets carries a model category structure whose weak equivalences are the quasiequivalences, the cofibrations are the injective maps, like the usual model structure which models classical homotopy theory, and the fibrant objects are the quasicategories. This model structure, introduced by Joyal, is sometimes called natural, to differentiate it from the usual model structure. The word natural refers to one of the most basic notions in category theory: natural transformations. The original source by Joyal on the existence of this model structure was never published and it is not available. A proof can be found in [6].

The 1-simplex is also an interval in the natural model structure on simplicial sets, hence quasiequivalences induce contravariantly bijections between sets of homotopy classes of maps with a quasicategory as target. All this transfers to the comma category of simplicial sets over $N\tau_1 Y$ in the usual way. Hence, this theorem would follow from Theorems 3.3.3 and 3.4.2. $\square$

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