Towards an Approach to Teachers' Professional Development: How to Work with Algebraic Thinking in the Early Years

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This article presents the first intervention cycle of a design-based research program on teacher professional development. The study aims to understand how the formative process helped teachers to understand what algebraic thinking means and how to work with it in the early years. Data analysis was based on three principles of design: teacher's role and actions, professional learning tasks for teachers, and discursive interactions among participants. The results suggest that these design principles contributed to teachers' understanding of the meaning of algebraic thinking and how to promote it in elementary students.

Keywords: Algebraic thinking; Discursive interactions; Early algebra; Professional development; Professional learning tasks; Teacher educator

Hacia un enfoque para el desarrollo profesional de los docentes: cómo trabajar con el pensamiento algebraico en los primeros años

Este artículo presenta el primer ciclo de intervención de una investigación basada en el diseño del desarrollo profesional docente. Su objetivo es comprender cómo el proceso formativo ayudó a los maestros a comprender qué significa el pensamiento algebraico y cómo trabajar con él en los primeros. El análisis de datos se basó en tres principios de diseño, papel y acciones del formador de docentes, tareas de aprendizaje profesional para los docentes e interacciones discursivas entre los participantes. Los resultados sugieren que los principios de diseño contribuyeron a la comprensión del significado del pensamiento algebraico y cómo promoverlo en los estudiantes de primaria.

Términos clave: Álgebra temprana; Desarrollo profesional; Formador de docentes; Interacciones discursivas; Pensamiento algebraico; Tareas de aprendizaje profesional

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The planning and implementation of professional development processes that actually contribute to developing teachers' knowledge is one of the challenges of education, particularly of mathematics education (Kazemi & Hubbard, 2008), considering that teachers in possession of mathematical knowledge for teaching are in a better position to foster student learning (Ball et al., 2005).

Several studies may currently be found that refer to a central set of characteristics of effective professional development processes (Desimone, 2009). However, there is little consensus as to "how" professional development processes can effectively improve teacher learning considering (i) the relationships established between their different components; and (ii) the immense variety of forms and contents of such processes (Kennedy, 2016). Thus, structuring professional development processes, in terms of their design and implementation, so that they contribute to the creation of professional learning opportunities for teachers (Ribeiro & Ponte, 2020) is a field of study that calls for further exploration (McDonald et al., 2013).

An intervention was designed, by means of a design-based research methodology (Cobb et al., 2016), with the aim of promoting professional learning opportunities for teachers in the early years. To this end, the present study considered the dynamics of the professional development process and a set of design principles, aiming to respond to the question: How did a professional development process help teachers to understand the meaning of algebraic thinking and how it develops in the early years of elementary school? This study aims to contribute to the construction of guiding principles for planning and conducting professional development processes, based on the characteristics of effective professional development processes (Desimone, 2009) and on a model geared towards designing and evaluating professional development processes (Ribeiro & Ponte, 2020).

THEORETICAL FRAMEWORK

High-Quality Professional Development

In order to ensure the high quality of a professional development process, five characteristics must be present (Darling-Hammond et al., 2017; Desimone, 2009; Fundação Carlos Chagas, 2017; Kennedy, 2016): (i) focus on content: focus on subject knowledge and on how students learn this content; (ii) active learning: teachers participate in the process of building knowledge and skills, as opposed to passive listening; (iii) coherence: what is fostered in professional development processes should be in line with the curriculum and school objectives, teachers' prior knowledge and beliefs and framed by the context in which teachers work; (iv) sustained duration: referring to the total amount of time attributed to meetings with continuous and intensive opportunities; and (v) collective participation: groups of teachers who participate and build an interactive learning community.

In addition to these features, there are several others: (i) the use of practice models (records), including examples of students' work, video analysis, among others (Ribeiro & Ponte, 2020), and (ii) feedback with expert support in order to assist the group discussions "so that teachers think, receive suggestions and make changes to their practices. Feedback and reflection help teachers move towards the expert visions of practice" (Darling-Hammond et al., 2017, p. vi). Considering the extensive review carried out on these studies of the effectiveness of professional development programs, Kennedy (2016), in addition to highlighting the importance of teachers' motivation to participate in these programs, also indicates that their effectiveness depends largely on the pedagogy adopted (how they are conducted).

PLOT Model

The Professional Learning Opportunities for Teachers (PLOT) model (Ribeiro & Ponte, 2020), illustrated in Figure 1, highlights the Role and Actions of the Teacher Educator (RATE), the Professional Learning Tasks for Teachers (PLTT) and the Discursive Interactions Among Participants (DIAP) as fields which comprehensively contribute to the creation of model, on5 the basis of a context.

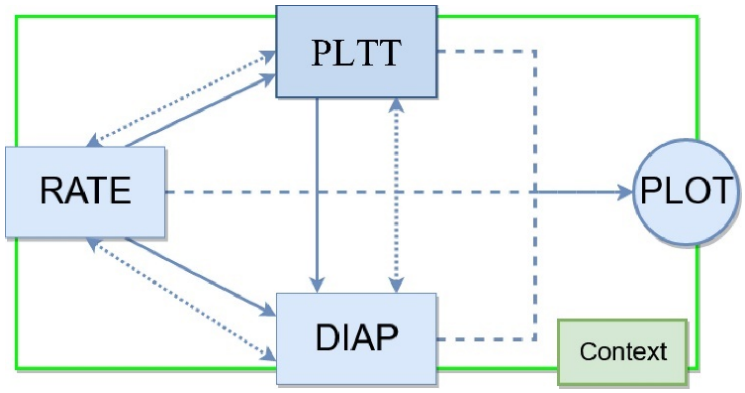


Figure 1. PLOT model (Adapted from Ribeiro and Ponte, 2020, p. 4)

By considering teacher learning as situated and mediated by instruments and context (Lave & Wenger, 1991), the PLOT model establishes parameters for the design, performance and evaluation of professional development processes, proposing that "by articulating these three domains in a single system, a theoretical and methodological tool is generated to organize and implement professional development processes that foster learning opportunities for teachers who teach mathematics" (Ribeiro & Ponte, 2020, p. 5).

Although not referring directly to high-quality professional development processes, the PLOT model brings many of its characteristics to light, as will be discussed later, when the core elements of each of the domains and a specific context of the importance of teaching algebra in early years are presented.

Role and Actions of the Teacher Educator (RATE)

Several studies have highlighted the skills required by the teacher educator, including "selecting and using appropriate tools and resources for teaching" (Zaslavsky, 2008, p. 98). When designing professional development processes to accommodate the characteristics of the local context (Desimone, 2009), the teacher educators consider mediation and management actions through exploratory teaching (Ponte, 2005; Ponte & Quaresma, 2016) in order to create professional learning opportunities (Ribeiro & Ponte, 2020). Based on the characteristics of effective professional development, the role of the teacher educator involves providing adequate and pertinent feedback for each situation (Jaworski & Huang, 2014), ensuring the collective participation of the participants (Desimone, 2009). The teacher educator's actions and questions should contribute to reflection, seeking to establish a relationship between the theory and teachers' experiences and practices (Ball & Cohen, 1999; Silver et al., 2007), in addition to promoting discussions on the proposed objectives for the formative process (Smith, 2001). In order to foster these discussions, not only should the teacher educators know the teachers with whom they will be working and their specific expectations, needs, and availability (Ponte et al., 2017), but they should also listen attentively and steer the process so that mathematical and didactical knowledge is duly articulated and orchestrated (Ribeiro & Ponte, 2020). According to Remillard and Geist (2002), the role of the teacher educator should involve "promoting opportunities for participants to build their own ideas about teaching" (p. 15). As far as teacher educators are concerned, the following four factors are necessary for these actions to be accomplished: (i) pedagogical preparation, (ii) subject content preparation, (iii) didactical knowledge of the content to be taught, and (iv) knowledge of the context (Vaillant, 2002).

Professional Learning Tasks for Teachers (PLTT)

A vast body of literature has stressed the centrality of the mathematical tasks presented to students to the success of their learning (Christiansen & Walther, 1986; Ponte, 2005). These tasks take on the form of a tool that "plays an unquestionable role in the opportunities for students' mathematical learning" (Delgado, 2014, p. 27). Likewise, PLTT transport many of the characteristics and functions of a mathematical task, thus being key to teachers' professional learning (Zaslavsky, 2008).

Professional learning tasks, situated within a scope in which practice is highly present, that inseparably focus on mathematical and didactical knowledge (Ribeiro & Ponte, 2020) may be defined as tasks "that engage teachers in the work of teaching, and can be designed to meet a specific teacher learning goal while taking into account the prior knowledge and experiences the teachers bring to the activity" (Smith, 2001, p. 8). By adopting the term "authentic practice samples", and also "records of practice" (Ribeiro & Ponte, 2020), Smith (2001) states that materials such as classroom videos (Borko et al., 2008), students' work (Kazemi & Franke,

2004) and high cognitive level mathematical tasks (Ribeiro & Ponte, 2020) portray teaching practice and can create opportunities for teachers to analyze and evaluate real classroom situations (Silver et al., 2007). In order to accomplish the material analysis and evaluation process, an exploratory teaching approach should be adopted (Ponte & Quaresma, 2016; Ribeiro & Ponte, 2020) which, by promoting the collective participation of teachers, seeks to establish a shift from the traditional isolation of teachers' work (Ball & Cohen, 1999).

Discursive Interactions Among Participants (DIAP)

Discursive interactions among participants are directly associated with collective participation (Darling-Hammond et al., 2017). It is assumed that professional learning opportunities can materialize from exchanges among peers, through dialogic communication (Ribeiro & Ponte, 2020) where teachers can "compare their theories, process new understandings, challenge deep-rooted beliefs, increase problem-solving skills, rebuild their practice, improve their self-confidence and strengthen their identity" (Fundação Carlos Chagas, 2017, p. 33).

The exploratory teaching approach fosters discursive interactions as it presupposes the circulation of ideas, experiences and mathematical and didactical knowledge among teachers (Ribeiro & Ponte, 2020). Some authors suggest that exploratory teaching should be carried out in four stages (Ponte et al., 2017; Stein et al., 2008): introduction, students' autonomous work, whole class discussion and systematization. When discussing adults' formative processes, it should be borne in mind that they "come to learning with experiences that should be used as a source for new learning" (Darling-Hammond et al., 2017, p. 7). Thus, the introduction stage, in which the PLTT are presented, and their purpose has been understood by all the participants (Ponte et al., 2013), is when teachers' prior knowledge is ascertained. The accomplishment stage involves working in small groups or individually to engage in the PLTT (Silver et al., 2007), taking the practice records into consideration (Smith, 2001). The whole class discussion period is the stage at which the solutions of the small groups are presented and discussed collectively, under the coordination of the teacher educator (Elliott, et al., 2009; Stein et al., 2008). Finally, the systematization stage is when the teacher educator seeks to establish relations between the elements emerging from the discursive interactions and the teachers' knowledge identified at the beginning of the PLTT, thus seeking to achieve their goals.

Algebra in the Early Years of Elementary School

When reflecting on algebra in the early years, the discussion should take into consideration what is referred to as algebraic thinking, since, as highlighted by Ponte et al. (2009), the main objective of the study of algebra in elementary and secondary education is "to develop algebraic thinking in students" (p. 10). Blanton (2008) defines algebraic thinking as "a way of thinking that permeates every dimension of mathematics and is the core of what children should be routinely

doing in school mathematics" (p. xii), taking into account the actions of "generalizing, expressing and justifying relationships between quantities, as well as reasoning with generalizations expressed through a variety of representations" (Blanton et al., 2011, p. 1). Thus, it is essential in algebra teaching that teachers understand the meaning of generalization (Ayala-Altamirano & Molina, 2021), which, according to Llinares (2018) "consists of moving from the particular to the general and of seeing the general in what is particular" (p. 54).

Moreover, when discussing the development of algebraic thinking in the early years, one is referring to a reformulation and deepening of arithmetic itself (Molina, 2009) and not to the addition of more content to the curriculum, since the main premise is that "the tasks to support algebraic thinking can be created directly from the materials in one's hands" (Blanton, 2008, p. xii).

The Teacher Education Process

Because early years teachers have little experience of the classroom practice that can support algebraic thinking (Hunter et al, 2018), and in view of the recent inclusion of this theme in the Brazilian national curriculum (Ferreira, 2017), the country in which this study was carried out, in-service teacher education is a cornerstone for creation of professional learning opportunities (Ribeiro & Ponte, 2020).

The professional development process, namely "Didactics of mathematics and the development of arithmetic and algebraic thinking", had 14 participating teachers who signed up on their own initiative, which shows their motivation to participate (Kennedy, 2016) and who taught in the 3rd, 4th or 5th years classes in the same educational network, and therefore belonged to the same context (Darling-Hammond et al., 2017). In addition, it had a duration of 64 hours, distributed in 32 face-to-face hours (8 weekly meetings of 4 hours, approximately 3 months of meetings) and 32 hours of autonomous work. The first author of this paper was the teacher educator.

The formative process designed by the authors adopted a situated perspective of learning (Lave & Wenger, 1991), geared towards teachers' understanding of the meaning of the work with algebraic thinking and its development in the early years. Its design and management were grounded in principles based on the teacher's teaching practice (Ball & Cohen, 1999): (i) the format of the professional development process was in line with the themes to be worked on with the students (Ponte et al., 2017); (ii) the use of curriculum documents (Brasil, 2017) and the textbooks used by the teachers (Passos & Nacarato, 2018); (iii) the teacher's work cycle in terms of planning, action and reflection (Smith, 2001); (iv) the professional development through the professional learning tasks (Silver et al., 2007); and (v) consideration of the characteristics of effective professional development (Desimone, 2009).

As presented in Tables 1 and 2, the PLTT carried out in the first and second meetings were of an introductory nature with the objective of leading the teachers to understand the need for working with algebra in the early years and seeking to find algebraic elements in the mathematical tasks already used by the teachers (Blanton, 2008).

The PLTT from the second meeting onwards were related not only to the mathematical content required for the development of algebraic thinking, generalized arithmetic, functional thinking and the different interpretations of the equal sign (Hohensee, 2015), but also to the didactical knowledge for their operationalization in the classroom.

Table 1
Relations among the goals, meetings and PLTT. Meetings 1st to 4th

Goals	1st meeting	2nd meeting	3rd meeting	4th meeting
1. To understand the need for working with algebra in the early years;		BNCC and the teaching of algebra		
2. To understand the relationship between arithmetic and algebra in the early years;	Analyzing a multiplication	BNCC and the teaching of algebra		
3. To understand what it means to work with algebra in the early years;			Generalization	
4. To understand how work with algebra is developed in the early years;		Mathematical tasks	Generalization	Reasoning and mathematical representation Teaching approach
5. To design a mathematical task for the development of algebraic thinking;		Planning a lesson		Planning a lesson
6. To manage and evaluate students' work in tasks involving algebraic thinking;			Reflection on a lesson	

Note: BNCC is the Brazilian National Curriculum.

Table 2
Relations among the goals, meetings and PLTT. Meetings 5th to 8th

Goals	5th meeting	6th meeting	7th meeting	8th meeting
1. To understand the need for working with algebra in the early years;				
2. To understand the relationship between arithmetic and algebra in the early years;				
3. To understand what it means to work with algebra in the early years;	Equal Sign	Algebraic skills in the BNCC Functional Thinking		
4. To understand how work with algebra is developed in the early years;	Equal Sign	Functional Thinking	Analyzing a mathematical lesson	
5. To design a mathematical task for the development of algebraic thinking;			Planning a lesson	
6. To manage and evaluate students' work in tasks involving algebraic thinking;	Reflection on a lesson			Reflection on a lesson

Considering professional learning as the result of activities carried out externally, but at the same time incorporated into the work of teaching and helping to change teachers' practice (Darling-Hammond et al., 2017), the Planning and Reflection of a lesson PLTT took the teachers' work cycle into consideration (Smith, 2001), through continuous action and analysis (Kazemi & Hubbard, 2008). These PLTT refer specifically to the teaching practice in which the teachers, through collective participation, plan a lesson and report on the mathematical and didactical knowledge involved in the professional development process so that, in the following meeting, a discussion on questions given at a prior stage could be established.

METHODOLOGY

This study followed a design-based research methodology —DBR (Cobb et al., 2016), which made it possible to approach a professional development situation by means of a design with specific professional development objectives. The entire professional development process was filmed by two cameras, one viewing the

teacher educator, and the other, the participants. The subgroup discussions were audio-recorded and written records of the teachers' work were collected.

One of the main features in DBR is the definition of design principles (Cobb et al., 2016). Considering the central aspects of effective professional development and the PLOT model domains, the following design principles were established: (i) the RATE, (ii) the PLTT, and (iii) the DIAP. Thus, the premise of the present study is that an intervention based on these principles fosters professional learning opportunities for teachers of the early years of elementary school, regarding the understanding of meaning and the development of work with algebraic thinking.

The data analysis was performed in three phases. In the first, by means of a retrospective analysis (Cobb et al., 2016), the videos, audios, documentation produced by the teachers, and the design of the professional development process were analyzed through a framework that considered two interconnected aspects: (i) refinement of the professional development process for the preparation of the following cycle; and (ii) the search for possible professional learning opportunities created by the professional development process, considering whether the PLTT design and management met the characteristics adopted by the PLOT model and the principles of effective professional development. A report of each meeting was produced on the basis of this analysis.

The second phase, observing the professional development process as a whole, analyzed the design of the categorized PLTT in terms of the following categories (i) the professional development process goals, (ii) the knowledge focus, and (iii) how the professional developing process was conducted.

The analysis in the third phase sought aspects from the reports (of the 1st phase) and categorizations (of the 2nd phase) referring to the two central themes of the professional development process: changes mentioned by the teachers related to algebraic thinking and the exploratory teaching approach. The specific work with algebraic thinking was operationalized through the *Generalization* PLTT and its counterparts, which aimed to work on the meaning of generalization and its importance for the work with algebra in the early years. This PLTT was carried out in the third meeting and resumed at the beginning of the fourth, where the teachers were also asked to design a plan that would gear the students towards generalization situations. The main objective of the exploratory teaching approach was for teachers to translate into practice how to develop algebraic thinking in the classroom. This was operationalized through two PLTT: the Teaching Approach PLTT (worked on in the fourth meeting, seeking to differentiate traditional from exploratory teaching, considering the role of the student and the teacher in each of the approaches) and the Analyzing a mathematics lesson PLTT, to further examine the characteristics of the phases of this approach.

RESULTS

Considering the changes mentioned by teachers and the ways in which the professional development process provided teachers with opportunities for professional learning (Ribeiro & Ponte, 2020), this section is divided into two parts. The first —Generalization PLTT and its outcomes— presents situations that include the specific work with the meaning of algebraic thinking. The second —Teaching Approach PLTT and its outcomes— presents situations that encompass the transposition to the practice of developing algebraic thinking in the classroom.

PLTT Generalization and its outcomes

In the first meeting of the professional development process, through the Analyzing a multiplication PLTT, in view of some of the students' productions (Figure 2) the teachers were asked to analyze the knowledge they conveyed and to identify the procedures they had adopted to solve the following task: "Pedro was really happy because he had finally managed to complete his fourth sticker album! Each album has 225 stickers. How many stickers does Pedro have in total?"

Following an analysis of this record of practice in the subgroups, during the whole group session the teacher educator asked the participants which of the five representations showed evidence of algebra, seeking to ascertain the teachers' knowledge of the main content of the professional development, which Débora indicated Bernardo's representation, justifying:

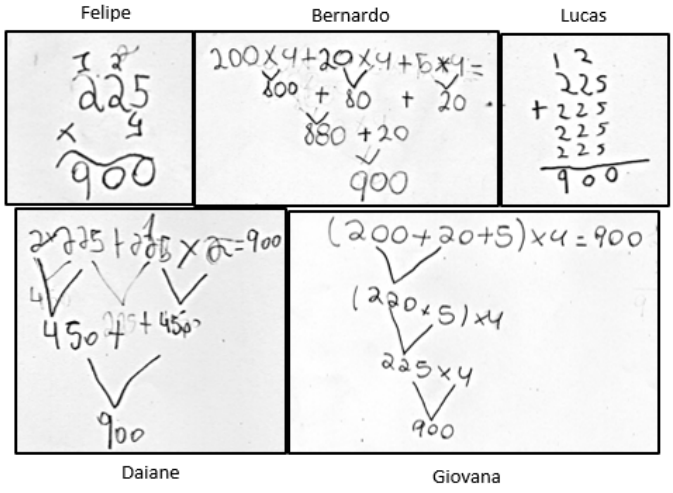


Figure 2. Record of practice of the Analyzing a multiplication PLTT

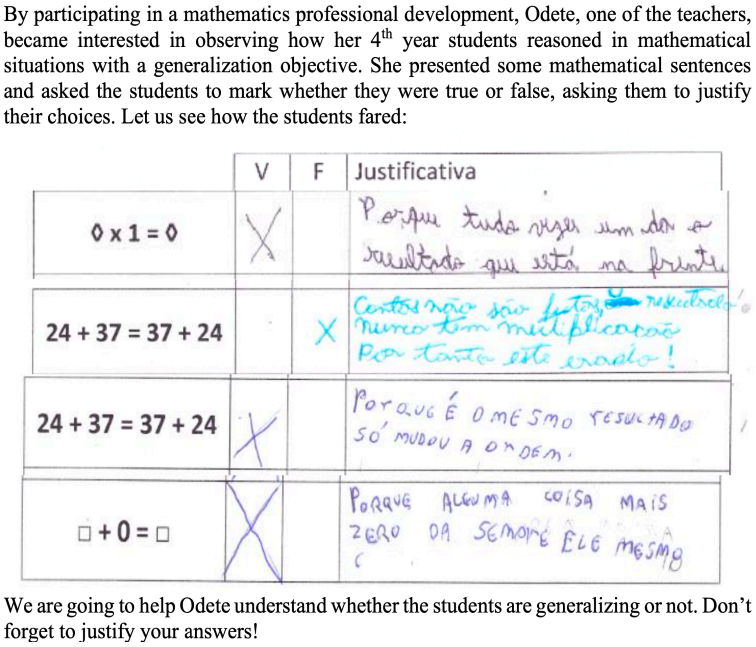
Débora: Because it shows that it's a numerical expression, there's more than one operation, so he's generalizing, he's come out of a specific calculation... Let's suppose that there [in Bernardo's representation] he used multiplication and there addition, then he's using more than one operation to get the result. I think this is related to algebra, it's not based solely on one principle or a single operation…

In this excerpt, Débora associates generalization with the presence of more than one operation, in addition to linking them to numerical expressions. After the work with the Generalization PLTT (in the 3rd meeting), in which the meaning of generalization, its importance for the development of algebraic thinking, and the way of working with students in the early years were discussed, Débora expressed her concept of generalization, associating it to patterns and regularities:

Teacher educator: What is generalization in mathematics?

Débora: It's when there's a situation, a certain concept that serves not only for the specific situation, but for several, there's a pattern, a regularity, but for more than one...for example when they understand the regularity of multiplication tables...

Considering that one of the objectives of algebra is to develop algebraic thinking, and that generalization is one of its pillars, it is possible to observe that after the Generalization PLTT, Débora gave a new meaning to generalization. The design of the Generalization PLTT focused on algebra and its teaching and, in line with most of the other PLTT, it consisted of 4 phases: (i) introduction to the topic by ascertaining the teachers' knowledge; (ii) analysis of students' productions in subgroups (as seen in Figure 3); (iii) whole group session, in which the teachers explained their reflections; and, (iv) systematization of the discussions, when the objective of the PLTT was recalled. In the subgroup discussion (accomplishment phase), records of practice were given (Figure 3), where the teachers were expected to analyze the students' justification regarding the veracity of a mathematical sentence.


By participating in a mathematics professional development, Odete, one of the teachers, became interested in observing how her 4th year students reasoned in mathematical situations with a generalization objective. She presented some mathematical sentences and asked the students to mark whether they were true or false, asking them to justify their choices. Let us see how the students fared:

	V	F	Justificativa
0 x 1 = 0	X		Porque tudo vezes um dá o resultado que está na frente.
24 + 37 = 37 + 24		X	Contas não são fatos numéricos tem multiplicação Por tanto este errado!
24 + 37 = 37 + 24	X		Porque é o mesmo resultado só mudou a ordem.
□ + 0 = □	X		Porque alguma coisa mais zero da sempre ele mesmo

We are going to help Odete understand whether the students are generalizing or not. Don't forget to justify your answers!

Figure 3. Record of practice of the Generalization PLTT

This part of the PLTT focused both on mathematical knowledge translated into generalizations and on the students' way of thinking, as the teachers needed to interpret and give meaning to the different justifications.

For the second mathematical sentence in Figure 3, concluded to be false by the student because the sums are not complete, the result never has a multiplication, some of the teachers made the following observations:

Moisés: There's no coherence...

Marina: They don't normally see a sum followed by something else. The teacher always gives the sum but doesn't give the result. They have to search for the result.

Adriana: Ah, I'd never thought of that...

As far as Adriana and Moisés were concerned, the student's answer made no sense, since the justification appeared strange to them. On the other hand, Marina identified that, in the student's view, after the equal sign there cannot be another sum, only a result, considering the operational perspective of the equal sign. This discussion led Adriana to look at the situation differently:

Adriana: I had never thought about what she [Marina] has just pointed out... in my mind the student simply hadn't understood [that] if I add 24 and 37 it's the same as adding 37 and 24. She didn't understand what was being asked there, that the result can't have sums...

Marina's experience and knowledge enabled her to identify in the student's answer that after the equal sign there cannot be another operation. By sharing her interpretation with the other teachers during the whole group session, Marina's contribution led them to look at the student's answer from a different perspective, thus fostering the emergence of a professional learning opportunity arising from discursive interactions among peers, on the basis of a collaborative process.

Also, in the Generalization PLTT, the teacher educator asked the teachers about steering the students towards generalization, emphasizing didactical knowledge at this point:

Teacher educator: To challenge a child, for example, if he says that 2 plus 3 equals 5 and 3 plus 2 equals 5, it's equal because the result is the same. How would you challenge a child so that he would reach an effective generalization? Because there he's only looking for the result. How would you do it?

Camila: With a large number

Teacher educator: That's right, so you would put 2345 plus 3782 is the same as 3782 plus 2345. And what does this show?

Débora: He'll see that only the position changes but the result won't change…

Estela: He won't manage to do the sum and will have to make observations, he'll have to look at the numbers.

In this discussion, it is possible to see that by means of an example given by the teacher educator requesting the teachers' thoughts on the classroom practice, they responded with a way for how to shift the student's reasoning from calculation to the behavior of the addition operation, for example, when Camila suggested using large numbers. While indicating how to make the students look at addition in a different way, without searching for the result, they presented arguments for why large numbers should be used, making it impossible to carry out the operation and thus leading the students towards the observation of the numbers and their relationships. Understanding how to get the students to look at the relationships and properties of numbers and operations is a fundamental part of working with algebraic thinking in the early years. Furthermore, by addressing the issue of how to lead students to generalize, associated with the action of the teacher educator where, instead of providing the answer, the teachers' reflections and positions were aroused, the PLTT contributed to the creation of professional learning opportunities.

Following the Generalization PLTT, Marina displayed a change in her practice:

Marina: ...starting to think algebraically, why is it only now that I have started to think algebraically with my students? Because I came to this course, so when we had that question of 10, 100 and 1000, I was able to make that reflection with them, thinking about this generalization, and in this case this was the direction I followed, but it's my way of thinking that has changed...

When Marina says, "because I came to this course", she shows evidence of having established a relationship between one of the contents normally worked on with the students, namely the arithmetic multiplication by 10, 100 and 1000 and one of the professional development topics (the regularities of mathematics). In other words, the teacher is able to relate what was worked on in the professional development sessions to her own teaching practice, showing her ability to observe mathematical situations in the classroom and the relationship established with the students' prior knowledge of arithmetic.

The Reflection PLTT, carried out at the 5th meeting, proposed a reflection in subgroups on the mathematical tasks that the teachers had developed with their students in the previous week. The math task planned by the 4th grade teachers contained ten division operations, four with a divisor equal to one, and the rest with other numbers in the divisor. In planning the mathematical task, these teachers sought to work on generalization, considering the difficulty of students with the operative technique of division (algorithm) in line with the fact that working algebra in the early years can be carried out from the arithmetic that has already been developed. in the classroom.

Débora: We think of the tables of the numbers that are in the keys [dividers], that any number multiplied by 1 will give the result that's in the dividend.

Teacher educator: But... only by the number 1?

Débora: Further ahead we can work on...

Teacher educator: I would put that on the board and ask the group. Look, this group thinks it's like this, do you all agree, does everyone agree? But only by the number 1? Why, for example, when I divide I had this number [quotient], if I multiply [by the divider] I have this number [dividend]. It's the same for any division.

Débora: I hadn't realized that.

In this excerpt, the teacher educator revealing the role of steering the problematizations, combined with her knowledge of the content, draws attention to the student's speech that had established a rule, which, although valid for number 1, would also be valid for other numbers, which Débora claims not to have noticed. Thus, the teacher educator's action of drawing attention to a mathematical regularity led the teacher to consider the division operation from a different angle.

PLTT Exploratory teaching approach and its outcomes

Regarding the introduction of a mathematical task, the teachers watched a video portraying a teacher introducing a mathematical task to his students and then read a text describing the importance and function of the introduction phase. Afterwards, the teacher educator asked the teachers how this exploratory teaching phase should be carried out:

Marina: The teacher needs to ask them if they have really understood, ask the child to explain what they need to do in the activity because we usually slip up. You can explore this in any activity, but you have to know what they have understood, because as they don't read [what they are expected to do], they just take the numbers and start their calculations. So, before they begin: I want to know what you need to do... and there the behavior starts to change... Since you [teacher educator] mentioned getting a student to talk [in an introduction] and explain, I do this on almost every occasion, even in other subjects and it's working out really well, because one reads and the other explains...

In this excerpt, Marina explains how mathematical tasks are routinely given. Regarding the teacher's actions, Marina states: "we usually slip up", showing that the teacher frequently fails to provide sufficient time and reflection for the students to interpret the context and mathematical situation contained in the task. On the other hand, the students do not stop to reflect and interpret either and, according to Marina, they end up not reading what they are expected to do, and simply arrange the numbers in order to create a sum.

The discussions carried out in the formative process, especially those in the Exploratory teaching PLTT, impacted the practice of Marina, since she claims to have adopted the phase of introducing the mathematical task with her students as witnessed in the formative process, thus demonstrating the teacher's role in questioning whether students have understood what is expected of them and the need for them to explain what they have understood.

In light of Marina's statements regarding the introduction of the task, the teacher educator asks another question:

Teacher educator:	Why are Marina's statements important?
Débora:	If they don't understand the instruction, they won't know where to begin.
Amanda:	Otherwise, they start [asking]: what are we supposed to do Miss?
Paula:	When they don't understand, they also lose interest. So when you ask them what they are expected to do, what they have understood, you are calling their attention. Paying attention to what they are reading usually arouses their curiosity and then they all understand and are able to carry out the task.

In this excerpt, the teacher educator's question, based on the position of one of the participants, produces a deepening of the discussion on the characteristics of the introduction of the mathematical task in which Paula addresses two central characteristics: making students understand what is expected of them (cognitive aspect) which, in turn, produces the interest and commitment required to complete the task (relational aspect).

Teacher educator:	Do you think that this format can help with the serious issue of the children not interpreting, not reading and not understanding the problem?

In this excerpt, in addition to fostering further discussion by means of a question, the teacher educator establishes a direct relationship with one of the difficulties faced by teachers in their daily practice, thus indicating her knowledge of the teachers' work context. This type of question suggests that a well-managed introduction phase can help students overcome the difficulty of interpreting the mathematical task, since the justification for students' difficulties in mathematics is commonly based on their deficient interpretation of the question, among other aspects.

In the Reflection PLTT in the 8th meeting, Camila showed how her students' reasoning had developed through her questions to a group of children during the completion of the mathematical task that had been planned in the previous meeting. The aim of this task was to work with functional thinking, where the students had to work out the number of wheels of a given set of cars.

Camila: Now it's time for the cars... [the pupils began to draw] in order to answer... they drew nine cars. Twelve, there was time [to draw]. And then I said: don't you think that's a lot of cars? Don't you think there may be another way of doing it? [the pupil responds] Yes, you just have to add. [the teacher asks] Just add? But how are you going to do the 20 cars? Are you going to add up the wheels? [the pupil responds]: Yes Miss, but... it's going in fours, isn't it? [the teacher replies] Yes. [the pupil asks] It's like the tables, isn't it Miss? [the teacher replies] Yes, it is like the tables. [the pupil speaks] so, we could stack them and put them into groups as we do in the tables... At this point I asked: How did you do the calculation? To solve the activities with the number 51, for example. Then the group came to the conclusion that it was the number of cars times four wheels. Then I went a little further: What if I were to write it in a way, to make a formula, so that it would work for any number... And then they wrote that the wheel is equal to c [cars] times 4.

This excerpt shows that, through the questions asked by Camila, the students were able to shift from the pictorial representation, drawing the number of cars and counting them (one by one) to the sum of four by four (by grouping) and, finally, arriving at a more sophisticated reasoning involving multiplication. It is important to note that this shift on the part of the students was only possible through the actions of the teacher who went beyond simply conveying a set of procedures to question and encourage reflection. The Analyzing a Mathematics Lesson PLTT had deepened the teacher's role and actions in Mathematics lessons, serving as a mediator of the students' learning. One of the discussions in the formative process for this exploratory teaching phase was related to the teacher's role in maintaining the cognitive challenge presented by the mathematical task, not giving answers, but leading the discussions:

Adriana: I think that socialization in orality is a key factor because you have reflection, but when you only have reflection per se, you keep it to yourself. Or if you have misunderstood something, you keep it to yourself and that's the end of it. When I stressed orality and that they had to talk [as well as] their peers, by talking in their own way, it was far more meaningful than if I had been talking at the blackboard. So, the key point of this activity is promoting reflection, group work, socializing, presenting arguments, communicating, something I didn't used to do, sometimes I would do the activity, ok now I'm going to correct it, at the most I'd ask a student to correct it on the blackboard...but I'd never shown the thinking and reflection…

In this report, Adriana explains her way of working with mathematics prior to the professional development. This consisted of presenting the students with tasks they had to do autonomously which were then corrected by the teacher, a very common format in classrooms in the early years. Thus, the way in which the PLTT was managed contributed to the creation of professional learning opportunities where

the teachers were able to understand other ways of working with mathematics, confirming the importance of the whole group session.

Regarding the role of the teacher educator, she gave the following feedback when confronted with a difficulty presented by Eliana:

Teacher educator:	What was the most difficult and easiest aspect and what would you do differently?
Eliana	The most difficult aspect of this activity in my opinion is reaching all the students.
Teacher educator:	How many children were in each group?
Eliana	4.
Teacher educator:	If you start putting them in pairs. An activity for two. Because in 4 one works, the other helps, the third pretends and the fourth doesn't pay the slightest bit of attention.

In this excerpt, considering the teacher's concern that all the students are able to effectively understand the mathematical ideas posed by the task, the procedure suggested by the teacher educator is that if the students are arranged in pairs, this might foster their commitment and mathematical understanding of the question, since in large groups the students with more difficulties might "get lost". The teacher educator's stance allowed for the provision of adequate feedback, based on her own understanding of didactical knowledge.

In the subgroup discussion of the 3rd year teachers, it was possible to observe that their exchanges of experience contributed to the emergence of a professional learning opportunity, based on Adriana's argument:

Adriana:	For example... I didn't separate the leaders, I left them all together.
Moisés:	I don't do it that way.. I get one who is above average, one in each group, usually one below average in each group so that it's a good mix…
Adriana:	But do you know what I'm afraid of, that the others [students] won't understand... that the other one [can do it] quickly and the others just... pretend they have understood…
Moisés:	They wouldn't have anyone to fall back on.
Adriana:	That's right, that's why I thought it best to leave the good ones together. I liked it because the students with difficulties managed to get there [complete the mathematical task], even with difficulties.

Based on the arguments presented by Adriana on the organization of groups with a similar level of mathematical understanding for the task, Moisés said he was going to change his practice:

Moisés:	What I'd do differently, I'd do what she did [Adriana grouped them] because I didn't do the same thing, I mixed a student who knows more, an average one and a below average one. But it's much better [Adriana's procedure]. I

> didn't have that perspective, but now I do. That's what I'm going to do when I form groups.

In this excerpt, the experience socialized by Adriana clearly contributed to Moisés' analysis of his own practice, reflecting a better way of forming groups with his students, thus showing that knowledge is mobilized in a context of interaction. In this case, the professional learning opportunity stemmed from the peer discussion in the Planning PLTT, which asked the teachers to indicate the criteria adopted in grouping the students to carry out the mathematical task.

Discussions of Results

Just as the role of the teacher is central to student learning (Jaworski & Huang, 2014), the teacher educator, similarly, plays an essential role in creating professional learning opportunities. It is incumbent upon the teacher educator to plan the professional development processes, to establish objectives, contents and strategies, duly organized into PLTT (Ribeiro & Ponte, 2020) and to orchestrate (Stein et al., 2008) the discursive interactions, triggering reflections and providing adequate feedback for each situation (Darling-Hammond et al., 2017). Thus, it was possible to perceive the supportive nature of the PLOT model domains for the emergence of professional learning opportunities, considering the teachers' references to changes both in relation to their understanding of the meaning of algebraic thinking and the way of working in the classroom.

In regards to the meaning of algebraic thinking, in its presentation of records of students' mathematical tasks (records of practice) (Ribeiro & Ponte, 2020) the Generalization PLTT required the teachers to collectively discuss (Desimone, 2009) the presence of generalizations in the students' justifications (Figure 2). In this case, the three domains of the PLOT model, the RATE when designing the PLTT, and the analysis of the records giving rise to DIAP may have contributed to Débora's redefinition of her understanding of the meaning of generalization. It is important for teachers to know the meaning of generalization situations, based on the arithmetic already performed, in order to propose mathematical tasks related to different mathematical regularities. Another example was when Marina noticed the relationship between content commonly worked on with students, namely multiplication by 10, 100 and 1000, and the work with regularities.

The design of Analyzing a Mathematics Lesson PLTT included a video (record of practice) portraying how a teacher performed the introduction of a mathematical task. By analyzing a concrete classroom situation (Ribeiro & Ponte, 2020), focusing on specific content (Kennedy, 2016), the teachers were able to understand the importance of the introduction for students' understanding and interest. This resulted not only from Marina's position, who stated that prior to the professional development process she had not given her students sufficient time to engage with the task, but also from that of the other teachers who presented the

characteristics of the introduction phase. The work carried out through this PLTT in conjunction with the discursive interactions (DIAP) managed to promote professional learning opportunities in association with the role of the teacher educator (RATE) who, in view of Marina's position, requested further explanations from the group, thus unveiling their knowledge of the content in question.

The way Camila managed to steer the mathematical task with her group of students may have been the result of the Analyzing a mathematical lesson PLTT. This PLTT fostered reflections on the teacher's role as mediator in the discussions, offering suggestions for the type of questions that could be put to the students to arouse their mathematical reasoning and avoid giving them ready-made answers. Here the integration of the domains of PLOT model was in the manner by which the teacher educator (RATE) prepared and steered the PLTT, evidencing the central role of a teacher educator in a professional development process.

During the discursive interactions (DIAP) of the Generalization PLTT, based on her knowledge and experience, Marina presented her interpretation that for the student there can be no other calculation after the equal sign, leading Adriana to change her initial interpretation. Although this reflection stemmed from the discursive interactions among the participants, the interdependent nature of the PLOT model dimensions was brought to light, as the presentation of a student record previously planned by the teacher educator (RATE) showed that the PLTT had been designed to arouse reflections on the student's mathematical reasoning, a feature under-explored by the PLOT model. In order to create professional learning opportunities, it is not sufficient to put teachers into discussion groups (Kazemi & Franke, 2004), it is also necessary to plan what will be discussed.

Several studies indicate that professional development programs should be based on practice (Ball & Cohen, 1999), leading teachers to build bridges between what is observed in the context of the professional development process and in their classroom. Thus, when considering the use of textbooks by the teachers in their daily activity, our professional development process may have contributed to the establishment of relationships between what is done in the professional development process and the teacher's practice. In addition to this external coherence (Desimone, 2009) between the elements of the professional development process and those of the teacher's daily life, it was possible to denote an internal coherence in the designed PLTT. Their circular format, consisting of an introduction, showing what teachers know about the content, accomplishment, discussion and systematization, the aim of which is to establish a relationship with the discussions and their initial knowledge, contributed to the emergence of professional learning opportunities.

Final Remarks

This study is based on three design principles, in line with the characteristics of effective professional development (Desimone, 2009). With regard to the role of the teacher educator (RATE) in both the development and management of the professional development process, it was found that her actions are inherently linked to her knowledge, as it was possible to note that many of the discussions steered by the teacher educator were related to her own knowledge of the context, particularly the didactical knowledge of the content to be taught. Regarding the various PLTT, they may be considered more robust when their design accommodates elements that foster discussions of a mathematical and didactical nature and include records of practice. Finally, the discursive interactions among the participants (DIAP) gave rise to the circulation of experiences and knowledge, and illustrated how the professional learning of teachers can stem from their interactive discussions within the scope of a process of knowledge reconstruction.

Another relevant aspect of the presented data is the interdependent action of the three design principles. Although it was possible to observe the most predominant design principle in each situation in the creation of professional learning opportunities, at a global level the inseparability of principles which are intrinsically connected to achieve the objective of the formative process was also observed. Thus, the design principles adopted by this study appear to provide teachers with professional learning opportunities aimed at bringing about changes in teachers' practice.

Thus, the design principles adopted by this study as well as the characteristics of effective high-quality professional development (Desimone, 2009) seem to contribute so that teachers in the early grades could understand what algebraic thinking means and how it develops in students. This study also shows that the model used to analyze the professional development (Ribeiro & Ponte, 2020) can help to design, conduct, and evaluate professional development, although more research should be carried out in order to qualify and deepen each of the domains.

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