Noticing students' mathematical thinking: characterization, development and contexts

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Mirar profesionalmente el pensamiento matemático de los estudiantes: caracterización, desarrollo y contextos

Resumen

Presentamos una síntesis de resultados de las investigaciones del grupo de Didáctica de la Matemática de la Universidad de Alicante realizadas durante los últimos años en relación a la competencia docente mirar profesionalmente. Las investigaciones se han centrado en tres focos: (i) caracterizar la relación entre las destrezas que configuran la competencia docente mirar profesionalmente; (ii) caracterizar grados de desarrollo e (iii) identificar características de los contextos que apoyan su desarrollo. Junto con los principales resultados, se identifican retos de futuro.

Palabras Clave. Mirar profesionalmente; destrezas; desarrollo; aprendizaje del profesor.

Olhe profissionalmente para o pensamento matemático dos estudantes: caracterização, desenvolvimento e contextos

Resumo

Este artigo apresenta uma síntese dos resultados da pesquisa do grupo de Didática da Matemática da Universidade de Alicante, realizada durante los últimos años em relação à competência de ensino para olhar profissionalmente. As investigações se concentraram em três enfoques: (i) caracterizar a relação entre as habilidades que compõem a competência de ensino para olhar profissionalmente; (ii) caracterizar graus de desenvolvimento e (iii) identificar características dos contextos que sustentam seu desenvolvimento. Os principais resultados são apresentados con os desafios para o futuro.

Palavras chave. Olhar profissionalmente; habilidades; desenvolvimento; aprendizagem do professorn.

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Abstract

We summarize results obtained by the Didactics of Mathematics research group at the University of Alicante on the competence of professional noticing. The research focused on three issues over the

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last years: (i) characterizing how the skills that make up professional noticing interrelate; (ii) characterizing different degrees of competence development; and (iii) identifying contexts that support this competence development. Main results are described along with future challenges.

Key words. Professional noticing; skills; development; teacher learning.

Regarder professionnellement à la pensée mathématique des étudiants: caractérisation, développement et contextes

Résumé

Nous présentons une synthèse des résultats des recherches du groupe de Didactique des Mathématiques de l'Université d'Alicante, réalisées au cours des dernières ans, sur la compétence enseignante regarder professionnellement. Les recherches ont fixé l'attention sur trois axes : (i) caractériser le rapport entre les habiletés donnant forme à la compétence enseignante regarder professionellement; (ii) caractériser le degré de développement et (iii) identifier les caractéristiques des contextes qui favorisent le développement. On présente des résultats avec des défis pour l'avenir.

Paroles clés. Regard professionnel; habiletés; développement; apprentissage du professeur.

1. Introduction

The teaching competence of *professional noticing of mathematics teaching-learning situacions* is considered to be a component of mathematics teachers' professional practice (Mason, 2002). The meaning of this teaching competence (professional noticing) is linked to how teachers use their knowledge of mathematics when performing different professional tasks such as: (i) selecting and designing tasks, (ii) analyzing and interpreting students' mathematical thinking and (iii) initiating and guiding mathematical discourse during class interactions (Figure 1, Llinares, 2013).

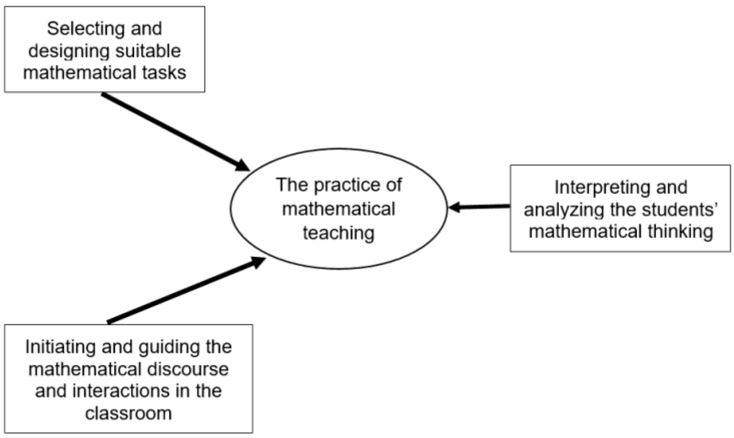


Figure 1: Activity system in mathematics teaching as a practice (Llinares, 2013, p.79)

Because of the relevance of this teaching competence, a line of research has emerged on its conceptualization and analysis during the initial training of primary and secondary school teachers of mathematics.

2. The teaching competence on professional noticing

Goodwin (1994) used the term *professional vision* to account for how professionals develop conceptual schemas to understand the situations they are in and that require decision-making. From this perspective, learning how to reason in a profession is considered part of building expertise.

In the case of primary and secondary school teachers, professionally noticing teaching-learning situations in mathematics should help them to: identify relevant aspects of learning situations; use their knowledge of mathematics and its didactics to interpret them; and make connections with more general principles on the teaching-learning of mathematics, in order to justify their teaching decisions (van Es & Sherin, 2002). This professional vision of mathematics teaching-learning situations implies switching from describing student or teacher actions, to focusing on how students are learning, and moving from evaluative comments to making interpretative comments based on evidence (Bartell, Webel, Bowen & Dyson, 2013).

In recent years, initial and continuing education programs, aimed at developing this teaching competence, have been providing information on the contexts and instruments that favour these transitions (Stahnke, Schoeler & Roesken-Winter, 2016), such as: the analysis of video clips showing interactions between teacher and students; or students solving problems (Llinares & Valls, 2009, 2010; Santagata & Yeh, 2016; Schack et al., 2013; van Es & Sherin, 2002; 2008); or the analysis of students' written answers to math problems focusing on errors (Son, 2013).

A particular aspect of this competence is to professionally notice students' mathematical thinking. Jacobs, Lamb and Philipp (2010) conceptualize the professional noticing of students' mathematical thinking competence as the interrelation between the three following skills: (i) identifying important mathematical elements in student responses; (ii) interpreting students' understanding of those mathematical elements; and (iii) deciding how to respond with this understanding in mind. Research developed in our group focuses on this particular aspect of the competence and centred on three specific areas:

- Characterizing the relationship between the skills linked to professional noticing of students' mathematical thinking.
- Characterizing degrees of development of this competence.
- Analyzing contexts that help to develop this competence.

The mathematical domains considered were: length magnitude and measurement, generalization of patterns, proportionality, limit of a function at a given point, derivative of a function and classification of quadrilaterals; and different teaching levels: pre-service preschool teachers, and pre-service primary and secondary school teachers. The first area has answered the question: how do pre-service teachers notice students' mathematical thinking (i.e., how do they identify, how do they interpret and how do they decide)? Results led us to understand relationships between the skills of identifying, interpreting and making decisions, during initial training programs while teaching competence developed. In the second area we have addressed the question: how do pre-service teachers develop this competence? Results led us to generate descriptors of degrees of development. In addition, our findings produced information on how pre-service teachers learned to use knowledge in practical situations. In the third area we answered the question: what contexts favour the development of this competence? We identified contexts that were able to enhance relations between the skills necessary for professional noticing of students' mathematical thinking.

3. Characterizing the relationship between skills

Jacobs et al. (2010) suggested that a nested relationship existed between the skills of identifying, interpreting and making decisions, in the sense that the ability to interpret depends on the ability to identify, and the decision on how to respond depends on the interpretation made. Based on this, identifying the important mathematical elements in students' strategies is a necessary condition to interpret students' understanding and decide how to respond (Barnhart & van Es, 2015). However, interpreting students' understanding and deciding how to respond, taking into account students' understanding, is a difficult task for pre-service teachers since their mathematical knowledge tends to be insufficient (Bartell et al., 2013; Son, 2013).

To support pre-service teachers' noticing of student strategies and procedures, with the aim of inferring characteristics of students' understanding, we used tasks that included a variety of student answers illustrating different degrees of understanding of a mathematical concept. In this way, pre-service teachers had to analyze and compare real student answers to one or several problems that revealed different characteristics of students' understandings. This aspect of our methodological approach constitutes a distinguishing feature of our research. Student answers to design tasks were selected based on Didactics of Mathematics research about the understanding of mathematical concepts in primary and secondary school students (Figure 2).

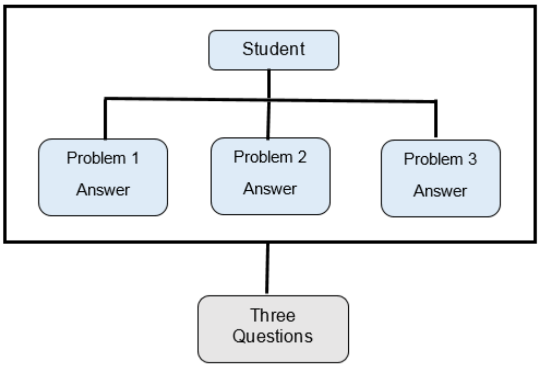


Figure 2. Task structure

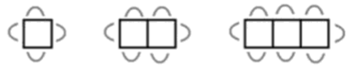
To analyze student responses, pre-service teachers had to answer three professional questions relating to professional noticing skills:

- Describe how student X solves each problem, indicating the elements of concept Y used and whether the procedure is adequate and why.
- Based on the description of how the student solves the problems, is it possible to identify some characteristic of how student X understands concept Y?
- If you were the teacher and considering how student X understood concept Y as reflected by his/her solution of the problem, what would you do to improve his/her understanding?

In Callejo and Zapatera (2017), the task was to analyze the responses of three primary school students to three problems related to pattern generalization. The problems provided the first components of a succession of figures that followed a pattern, together with questions relating to near, far and algebraic generalization. Figure 3 shows one of the problems and Figure 4 the selected answers to this problem of three primary school students. They reflect different degrees of understanding of the linear patterns. The pre-service teachers had to answer all three professional questions.

Problem 3

Observe the following figures representing tables and chairs:



1 table	2 tables	3 tables
4 chairs	6 chairs	8 chairs

As you can see, we have put 4 chairs around one table, 6 chairs around two tables, and 8 chairs around three tables.

1. Can you draw 4 tables and the number of chairs it should have?
2. How many chairs can we put around 5 tables in this way? And around 6 tables?
3. For a party we put 18 tables together along with the appropriate number of chairs. How many guests will be able to sit? Explain how you found your answer.
4. If there are 42 children invited to a birthday party, how many tables will we need to put together in a row? Explain how you found your answer.
5. Explain in your own words a rule connecting the number of tables and the number of chairs.

Figure 3. Problem 3 in Callejo & Zapatera (2017, p.316)

Mathematically, this problem includes three elements: (i) a spatial (distribution of figure elements) and numerical (number of elements that make up the figures) structure; (ii) the functional relationship (relationship between the place occupied by each figure and the number of elements that make it up); and (iii) the inverse functional relationship. To solve this problem, students had to coordinate the spatial and numerical structures, establish the functional relationship and its inverse relation.

Student responses were selected according to degrees of understanding of linear patterns according to Radford (2014). Student A's response showed that he/she was not coordinating the spatial and numerical structures (grade 1). Student B's response indicated that he/she was able to coordinate the spatial and numerical structure for near and far terms and to establish a functional relationship verbally, but was not able to reverse the process, that is, student B did not calculate the number of tables from the number of chairs (grade 2). Student C's answer indicated that he/she was able to establish a functional relationship (relationship between the position of the figure in the sequence and the number of elements that compose it) and its inverse relation (number of elements and number of the figure) (grade 3).

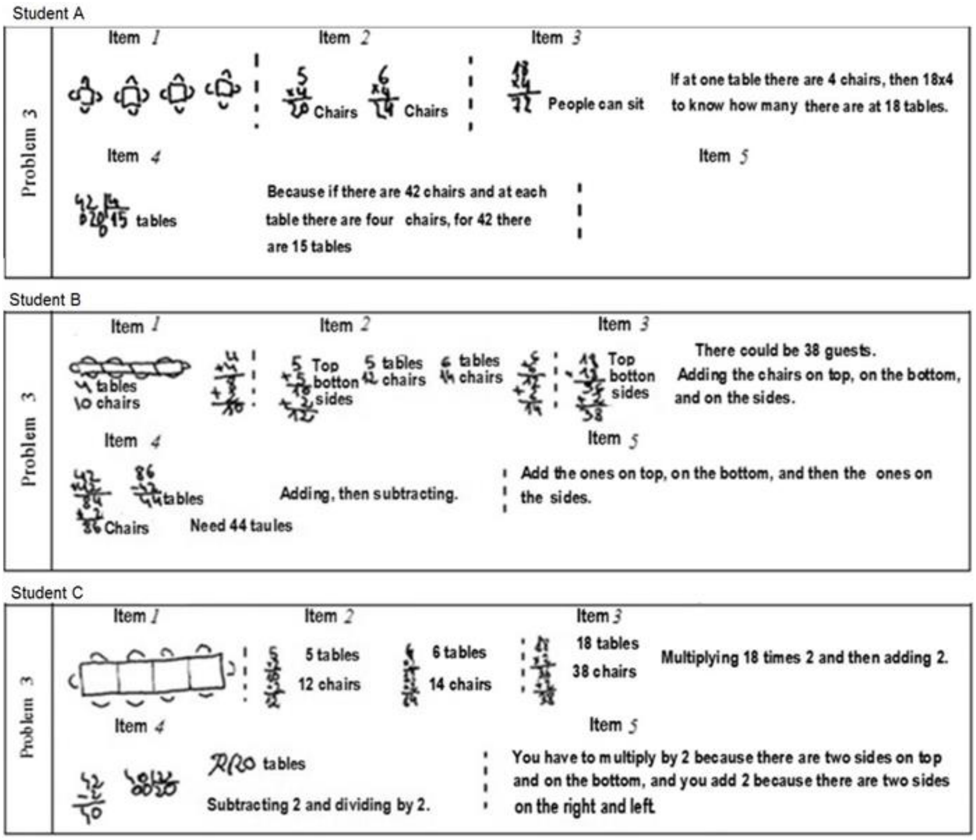


Figure 4. Student responses to problem 3

Our research on interrelations between the skills of identifying, interpreting and making teaching decisions, have taken into account different mathematical domains and levels of teaching: generalization of patterns in pre-service primary teachers (Callejo & Zapatera, 2017), proportionality in pre-service primary teachers (Buforn, Llinares & Fernández, 2018; Fernández, Llinares & Valls, 2013), limit of a function in pre-service secondary teachers (Fernández, Sánchez-Matamoros, Moreno & Callejo, 2018), classification of quadrilaterals in pre-service secondary teachers (Fernández, Sánchez-Matamoros & Llinares, 2015) and derivative in pre-service school teachers (Sánchez-Matamoros, Fernández, Valls, García & Llinares, 2012).

These studies generated findings on the relationships between the skills of identifying, interpreting and deciding in different mathematical domains. Callejo and Zapatera (2017) identified five profiles according to how pre-service teachers identify and interpret in the domain of pattern generalization:

- Profile 1: those who identify one or more mathematical elements in student responses, but do not recognize degrees of understanding of pattern generalization.
- Profile 2: those who identify at least one mathematical element and recognize evidence of near generalization (grade 1 of understanding).
- Profile 3: those who identify at least two mathematical elements and recognize evidence of near and far generalization (grades 1 and 2 of understanding).
- Profile 4: those who identify at least two mathematical elements and recognize evidence of near generalization and far generalization with inverse relation (grades 1 and 3 of understanding).

- Profile 5: those who identify the three mathematical elements and recognize evidence of the three grades of understanding of pattern generalization.

These profiles indicate a relationship between identifying relevant mathematical elements and interpreting students' understanding; however, a clear relationship between these skills and decision making was not found.

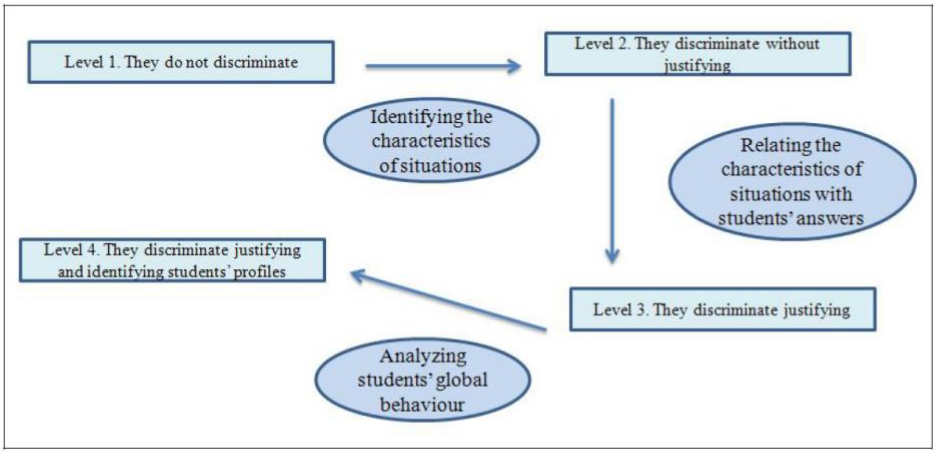


Figure 5. Indicators of how pre-service teachers identify and interpret mathematical thinking in students' proportional reasoning (Fernández et al., 2013, p.459)

Fernández et al. (2013) characterized how pre-service primary teachers identify and interpret students' mathematical thinking in the domain of proportional reasoning at four levels (Figure 5). Transition from level 1 to 2 corresponds to the ability to analyze a problem's features (mathematical elements) and thus differentiate them; transition from level 2 to 3 is linked to the ability to relate students' strategies with the problem's mathematical features, in order to justify whether the strategy is correct; and the transition from level 3 to 4 reflects the ability to analyze students' global behaviour (understanding) with regard to a type of problem. In this way, we created indicators of how pre-service teachers related the skills of identifying and interpreting.

In their study of pre-service secondary school teachers working in the domain of the derivative, Sánchez-Matamoros et al. (2012), showed the relationship between how pre-service teachers identified mathematical elements in the answers of high school students (low, medium and high level) and the way in which they interpreted these answers as evidence of students' understanding: some pre-service teachers provided general comments; other pre-service teachers only indicated if the student understood the concept or not; and other pre-service teachers recognized the characteristics of students' understanding (Figure 6). In addition, results showed that the teaching decisions proposed by pre-service secondary school teachers to support students' understanding of the derivative concept were related to how they interpreted their students' understandings. Thus, the pre-service teachers who identified characteristics of students' understanding proposed teaching decisions that focused on the meaning of the mathematical concept (conceptual actions).

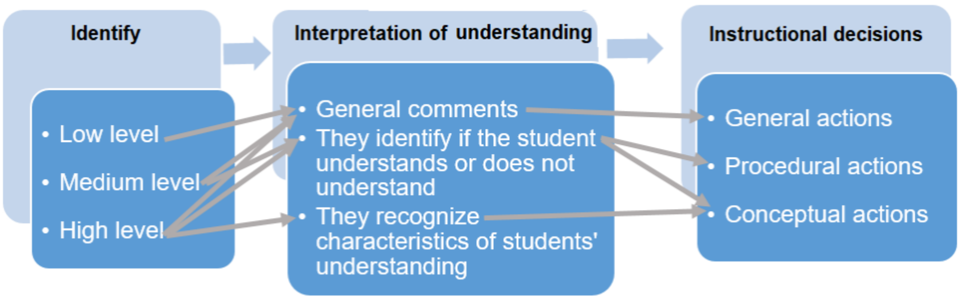


Figure 6. Interrelations between skills

3.1. Discussion and future perspectives

Results of our research indicate that the relationships between identifying, interpreting and making decisions are complex. However, there is evidence to suggest that they can be developed in initial teacher training programs.

Although there is a clear relation between the skills of identifying and interpreting, there is no direct relation with the ability to decide, which is difficult to acquire in training programmes (Choy, 2016; Jacobs et al., 2010; Santagata & Yeh, 2016). This is shown by the fact that pre-service teachers usually focus their decisions on general teaching procedures without considering students' conceptual progress. In our study on the derivative (Sánchez-Matamoros et al., 2012), only pre-service teachers who were able to identify different degrees of students' understanding of the concept of derivative proposed conceptual actions focused on students' understandings.

Deciding how to respond based on students' understanding is not easy. However, there is evidence to suggest that providing pre-service teachers with a frame or guide, enabling them to focus on relevant mathematical aspects in student responses could help them focus their noticing. In this sense, hypothetical learning trajectories (Simon, 2006) can be used by pre-service teachers as a guide to structure their attention to students' mathematical thinking (Edgington, Wilson, Sztajn & Webb, 2016). This would provide them with a mathematical and didactic of mathematics language helping to describe students' mathematical thinking. Hypothetical learning trajectories can help them to define learning objectives for each student, anticipate and interpret their students' mathematical thinking and respond, with appropriate instruction, to their progression in learning (Sztajn, Confrey, Wilson & Edgington, 2012).

In Ivars, Fernández and Llinares (2016), the use of a hypothetical learning trajectory of the fraction concept, provided to pre-service teachers as a theoretical document to support their noticing (designed based on Battista, 2012), endowed them with a specific language to refer to students' understanding, allowing them to focus on the relevant aspects of their strategies. In this way, using hypothetical learning trajectories as a guide to focus pre-service teacher attention seems to strengthen the relationships between identifying and interpreting, and interpreting and proposing activities (deciding). As a result of this research, hypothetical learning trajectories were included in the design of teaching modules of initial training programmes focusing on the development of competence with the aim of characterizing different degrees of competence development (our second research area).

4. Characterizing degrees of competence development

These studies were based on the methodology of teaching experiments, which interrelates teacher training practice and teacher learning research (Llinares, 2014). A teaching experiment includes research cycles taking place over three phases (Design-Based Researcher Collective, 2003). Phase 1 corresponds to the design and planning of instruction where the learning objectives, which define the goals to be achieved, are set and tasks are designed to facilitate the achievement of these objectives (teaching modules). Phase 2 corresponds to the implementation of the designed tasks. In phase 3, a retrospective analysis is carried out in which the teachers and researchers observe and analyze the experience, supporting their analyses with theoretical references. This analysis may lead to modifications in the tasks proposed to students (redesign).

Teaching (or instruction) modules are composed of sessions in which theoretical documents are provided (based on research in Didactics of Mathematics, in the form of hypothetical learning trajectories) as well as professional tasks. These professional tasks consist of teaching-learning situations (records of practice including student responses to problems illustrating levels of students' understanding, interactions between students and teacher, etc.) and three professional questions linked to the three teaching competence skills (identifying, interpreting and making decisions).

To illustrate the structure of these teaching modules, a module on the subject of derivatives, designed for pre-service secondary school teachers (Sánchez-Matamoros, Fernández & Llinares, 2015) is described below.

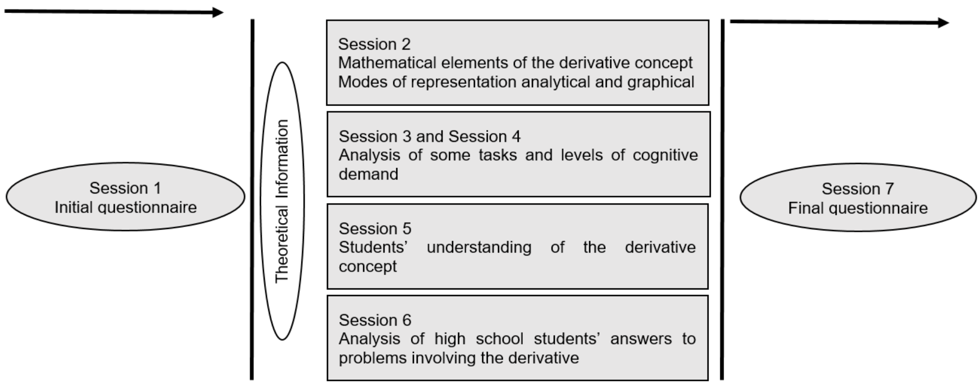


Figure 7. Module design outline in Sánchez-Matamoros et al. (2015, p. 1309)

The module consists of seven 120-minute sessions (Figure 7). In the first and last session, pre-service teachers carried out professional tasks aimed at understanding the development of their ability to recognize evidence of high school students' understanding of the concept of derivative. In the other sessions, the mathematical elements included in the concept of derivative in the different modes of representation, the cognitive demand of the problems, and the characteristics of high school students' understanding of the derivative concept were presented in the form of a hypothetical learning trajectory. This provided information allowed to pre-service teachers sharing a professional discourse on mathematical contents and on high school student understandings of the concept of derivative.

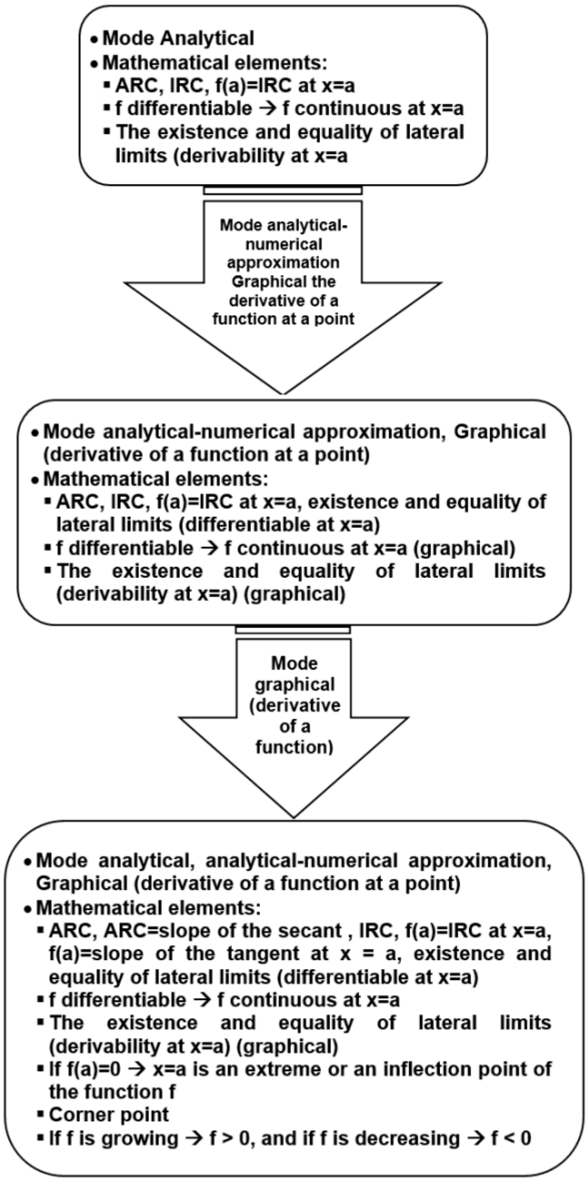


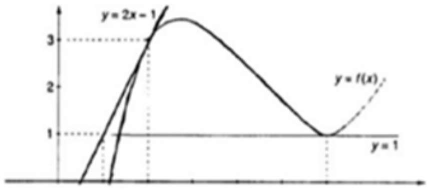
Figure 8. Progression in the understanding of the concept of derivative in Sánchez-Matamoros et al. (2015, p 1308)

This theoretical content, provided as a hypothetical learning trajectory of the derivative concept (a theoretical document), shows how the understanding of the derivative concept develops, based on findings by Sánchez-Matamoros, García and Llinares (2008) (Figure 8). It also presents a typology of the problems and answers proper to high school students. The contents of this hypothetical learning trajectory should be used by pre-service tachers to reason on the records of practice (responses of high school students to the problems of derivatives of a function).

In the study, the two professional tasks corresponded to sessions one (initial) and seven (final). They included the responses of three high school students to three problems (initial task) and another two problems (final task) (Figure 9), as well as three questions addressing the three professional skills. High school student answers revealed different characteristics of the understanding of the concept of derivative, in relation to mathematical elements and modes of representation, in accordance with the progression in understanding (Figure 8).

Problem 2

L is the tangent line to the graph of the function at a point (2, 3) as it is shown in the figure. Find f (2) and f ' (2).



Student 1		
Student answer		**Interview**
Answer	**Explain your answer**	**I:** You substitute in y=2x+1 to find the value of f (2). Could you find this value using another way? **S1:** Yes, using the graph. At x=2, the value of y is 3. **I:** f ' (2) = 2? **S1:** The derivative of "2x + 1" is 2. **I:** In the task, you have to find f ' (2) considering f(x) but you do not know its analytical expression. Could you reason it in another way? **S1:** I couldn't
$f(x) = 2x-1$ $f(2) = 2\cdot 2-1 = 3$ $f'(2) = 2$	*The value of f at a (2, 3) is y=2x+1* *If you substitute the x for 2, you obtain f(2)* *The derivative of f(2) is 2 because the function is 2x – 1* *Taking into account the graph when x is 2, the value of y is 3*	

Figure 9. Response of a student to problem 2 in a professional task (Sánchez-Matamoros et al., 2015, p. 1312)

We have considered different mathematical domains and teaching levels: pre-service preschool teachers, length and length measurement (Sánchez-Matamoros, Moreno, Callejo, Pérez-Tyteca & Valls, 2017); pre-service primary teachers and fractions (Ivars, Fernández & Llinares, 2017), pre-service secondary teachers and classification of quadrilaterals (Llinares, Fernández & Sánchez-Matamoros, 2016), the concept of derivative (Sánchez-Matamoros et al., 2015) and the concept of limit of a function (Fernández, Sánchez-Matamoros, Callejo & Moreno, 2015). Results allowed us to characterize degrees in the development of teaching competence. Descriptors of these degrees were based on: instrumentation of the hypothetical learning trajectory given in the teaching module (Verillon & Rabardel, 1995); progress in mathematical discourse describing student mathematical thinking provided in the hypothetical learning trajectory; and consideration of the understanding of certain mathematical elements of the concept as Key Developmental Understanding (KDU, Simon, 2006).

Sánchez-Matamoros et al. (2017) used the instrumentation of the hypothetical learning trajectory (Verillon & Rabardel, 1995) to analyze pre-service pre-school teachers' competence development in the domain of length magnitude and its measurement. The instrumentation of a hypothetical learning trajectory is understood as the way in which pre-service teachers construct their schemas of instrumental action in order to understand the trajectory's possibilities and limitations in assisting their reasoning in practice. These instrumental action schemes progressively turn into techniques leading to effective responses to professional tasks. This study shows that learning to use a hypothetical learning trajectory as a conceptual instrument did not take place directly, but was progressive.

We consider that the instrumentation, by pre-service teachers, of a hypothetical learning trajectory for a mathematical concept is evidence that they are developing their professional noticing of children's mathematical thinking. This working hypothesis is based on the idea that the development of instrumental action schemas is based on the implicit recognition of characteristics of children's understandings: they identify the mathematical elements involved in the teaching-learning situation and interpret children's answers, leading to propose tasks directed at children's progress in learning. Below are stages in the development of trajectory instrumentation:

- does not use the hypothetical learning trajectory as a conceptual instrument, by not identifying or rhetorically (meaningless) using mathematical elements;
- partially instruments the trajectory, when interpreting the understanding and set activities that allow some children to progress in their learning, but not all;
- instruments the trajectory when interpreting the understanding of all children, making use of the progression model and when proposing activities for all children, taking into account the progression model and the trajectory's set of tasks. This last level can be understood as the development of two schemes of instrumental action: one linked to the interpretation of understanding and another to proposing activities in accordance with that understanding.

The progressive use of the hypothetical learning trajectory throughout the teaching module generates indicators on how the competence of professional noticing of children's mathematical thinking develops (Figure 10).

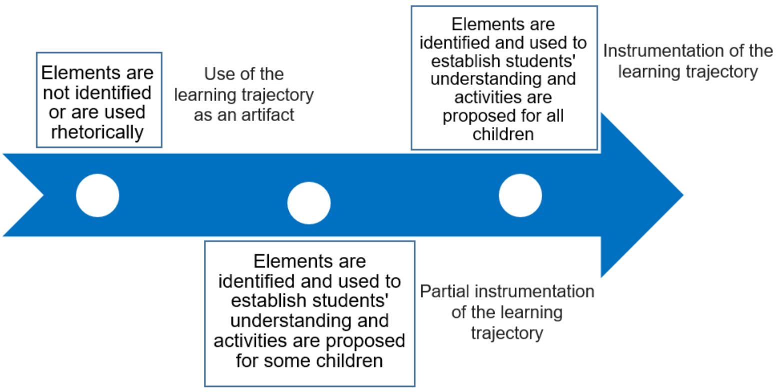


Figure 10. Characteristics of the development of professional noticing of preschool children's mathematical thinking on length and length measurement

Since the use of theoretical references in the teaching modules can provide pre-service teachers with a mathematical language and a language about students' learning enabling to describe students' mathematical thinking (Walkoe, 2015), Ivars et al. (2017) examine how progress in professional discourse may lead to indicators on the development of teacher's professional noticing of students' mathematical thinking. This research focuses on pre-service primary school teachers taking part in a teaching module designed around a hypothetical learning trajectory of the fraction concept. The initial results showed that, after participating in the module, pre-service teachers elaborated a more detailed discourse on students' mathematical thinking, providing evidence to support their interpretations. This led us to supose that the hypothetical learning trajectory helped to advance professional discourse. In addition, pre-service teachers who elaborated a more detailed discourse on students' mathematical thinking provided more decisions that did focus on students' conceptual progress. This suggests that progress in professional discourse seems to influence the ability to decide, generating activities that are more in line with students' understandings.

When hypothetical learning trajectories are introduced as a reference to help pre-service teachers interpret student mathematical thinking, an important aspect in the development of teaching competence is the recognition of the mathematical elements whose understanding advances student conceptual learning. Some studies have attempted to characterize degrees of development based on this aspect. Llinares et al. (2016), Sánchez-Matamoros et al. (2015) and Fernández et al. (2015) studied how pre-service secondary teachers considered the understanding of some mathematical elements of the concept as a Key Developmental Understanding (KDU). A KDU implies students advance conceptually, that is, they undergo a change in their ability to think and/or perceive mathematical relationships (Simon, 2006, p. 362). Thus, a KDU becomes a key element in the development of a concept whose recognition by pre-service teachers allows taking into account degrees of development of the teaching competence. The fact that pre-service teachers recognize the understanding of certain mathematical elements as a KDU can help them understand how students give meaning to mathematical concepts. Our assumption is that, if pre-service teachers focus their attention on the KDU of a mathematical concept, they will be able to better anticipate or interpret the development of the concept's understanding.

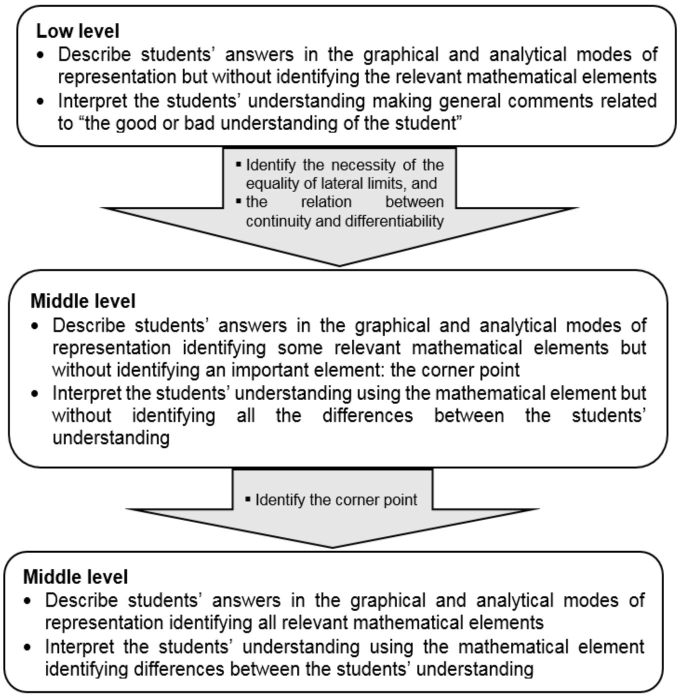


Figure 11. Degrees of development of the competence in the derivative domain (Sánchez-Matamoros et al., 2015, p. 1325)

Sánchez-Matamoros et al. (2015) provided descriptors of degrees of competence development in the domain of the derivative (low, medium and high) taking this into account. That is, degrees of development were determined by how pre-service secondary teachers considered the understanding of some mathematical elements of the derivative concept as KDU (Figure 11): (a) relation between the difference quotient limit and the meaning of the derivative as gradient of the tangent line, (b) relation between the derivability of the function and its continuity, and (c) information about the function or derivative function around the inflection points and the corner point.

In the same way, the results of Llinares et al. (2016) are based on the idea that students' understanding of inclusive relationships in the quadrilateral classification can be considered a KDU, since it requires a change in the student's ability to think and/or perceive relationships. Llinares et al. (2016) identified three changes in how pre-service teachers considered inclusive relationships in the classification of quadrilaterals as a KDU throughout the teaching module (Figure 12). These changes enable to characterize different degrees of development in this teaching competence.

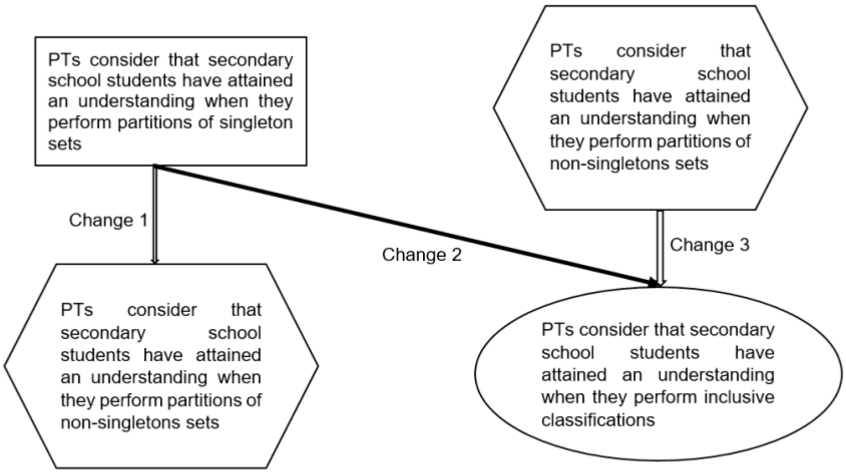


Figure 12. Changes in the recognition of inclusive relationships as a KDU (Llinares et al., 2016, p. 2163)

At the end of the teaching module, some pre-service secondary school teachers were aware that the ability of a secondary student to establish inclusion relationships between figures within a set was evidence of the understanding of quadrilateral classification (changes 2 and 3). In this way, these pre-service teachers began to use students' understanding of inclusive relationships as an indicator of the conceptual understanding (KDU). However, one group of pre-service teachers considered that this understanding was shown by the ability to create partitions of the set of quadrilaterals, assuming the existence of non-unit sets, though without progressing towards the recognition of the relationships between the properties that would generate some type of inclusive classification (change 1). That is, the pre-service teachers in this group did not recognize the role that the understanding of inclusive relationships could play in the learning of quadrilateral classification. These three changes made it possible to identify transitions in pre-service teachers when they were learning about secondary school students' understanding of the quadrilateral classification process. These results allowed us to infer characteristics on the different degrees of development of the teaching competence. These can be understood as characteristics of a learning trajectory for pre-service teachers (Figure 13).

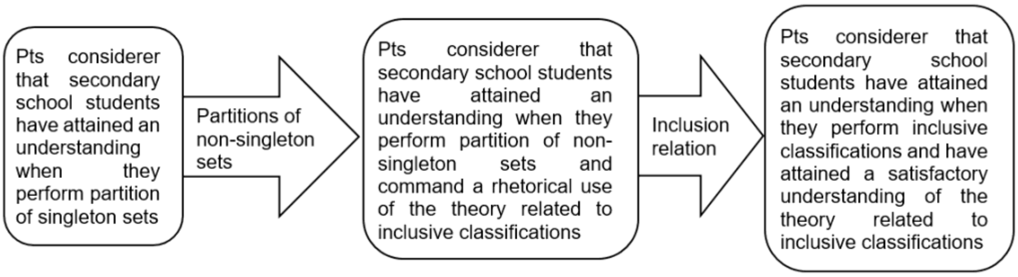


Figure 13. Pre-service teachers' development of noticing students' mathematical thinking relating to the quadrilateral classification process (Llinares et al., 2016, p. 2168)

Fernández et al. (2015) show that when pre-service secondary school teachers consider as a KDU, the understanding of approximation processes coordination across different modes of representation -when learning about the concept of limit of a function at a given point-, they could interpret different characteristics of understanding in the secondary students' responses. They could also propose decisions directed towards students' conceptual progress. Throughout the teaching module, three manifestations of competence development were observed. One manifestation was making incorrect or rhetorical (meaningless) use of mathematical elements of the concept of limit of a function at a given point, when describing the students' responses. This led them to simply recognize the answers as correct or incorrect. Another manifestation was to consider that students progressed in their learning when identifying characteristics of students' unsderstanding as they considered that the understanding of approximation coordination of domain and range across modes of representation was a KDU. The last manifestation consisted in supporting the proposal of new activities defining the understanding of approximation coordination across different modes of representation (the KDU) as a learning objective.

4.1. Discussion and future perspectives

Our research has generated descriptors of degrees of competence development in different mathematical domains. These descriptors inform on how competence begins to develop in initial training programs and can be used in the instructional design of training proposals (Sánchez-Matamoros et al., 2015; Llinares et al., 2016). However, more studies are needed in this line of research in order to design teaching modules based on empirically generated competence development trajectories.

As teacher educators and researchers, our aim is to explain how pre-service teachers learn to interpret teaching situations. The challenge is defining how to analyse the learning of this competence. Our research has contributed to characterizing pre-service teacher learning considering: the instrumentation of the hypothetical learning trajectory given in the teaching module; the progression in professional discourse on students' mathematical thinking; and the role of recognizing the understanding of certain mathematical elements of a concept as KDU.

Teaching experiments constituted the contexts for interrelating teacher training and research on teacher learning (Llinares, 2014). This methodology allows improving teacher training practice based on the design of teaching modules for teacher training programs, and on iterations of design and revision cycles. The question thus arises as to which teaching module features foster the development of teachers' professional noticing of students' mathematical thinking in training programs. Our research concludes by suggesting some of these features, as well as the design of professional tasks and contexts (third area of research). It also recommends actions that pre-service teachers should undertake to support the development of their teaching competence.

5. Contexts for competence development

We used different contexts to support the acquisition of skills constituting teaching competence. In previous sections we described some of the tasks that were carried out. In this section we specify other contexts that we used to support skill development: the writing of narratives during periods of observation of practices focusing on relevant learning situations of mathematics (Ivars et al., 2016); online discussions between pre-service teachers (Fernández, Llinares & Valls, 2012); interactions between in-service teachers and trainers (Coles, Fernández & Brown, 2013); and *feedback* between tutor and pre-service teacher (Ivars & Fernández, 2018).

5.1. Writing narratives

Narratives are stories in which the author sequentially relates events that make sense to him, based on an internal logic (Chapman, 2008; Ponte, Segurado & Oliveira, 2003). Considering pre-service teachers as storytellers in training programs can help them to observe teaching situations in an increasingly structured way, giving meaning to their experience during their practices.

In Ivars et al. (2016), a total of 39 pre-service teachers were asked, during their practices period in primary schools, to write a narrative describing classroom events that they considered potentially relevant to explain the mathematical learning they were observing. To articulate the narrative, they were given some guiding questions based on the three skills involved in professional noticing of teaching-learning situations: describe, interpret and decide (Figure 14). In their narratives, 21 out of 39 pre-service teachers identified the relevant mathematical elements of the teaching-learning situation, providing interpretations and evidence of students' understanding. However, only 11 out of these 21 pre-service teachers proposed decisions aimed at fostering student understanding of the relevant mathematical elements of the situation. On the one hand, these results indicate the potential of narratives to help structure observations of teaching situations. On the other hand, the narratives about teaching situations observed by pre-service teachers inform us on how teaching competence skills are integrated; the ability to make teaching decisions is especially difficult.

- Describe *"in detail" the mathematics teaching-learning situation.* The task (curricula contents, materials, resources…). What did elementary school students do? For example, you can indicate some students' answers to the task, difficulties… What did the teacher do? For instance, you can describe the methodology and some aspects of the interactions.
- *Interpret the situations.* Indicate the mathematical objective of the task and how the task was implemented. Indicate some evidence of students' answers that show that elementary school students had achieved the objectives (students' understanding of the mathematical content) and/or the difficulties they had.
- *Complete the situation.* Complete the situation indicating how you will continue in order to help students develop other aspects of the mathematical competence identified (that is, in order to support students in their conceptual development).

Figure 14. Guiding questions based on the three skills involved in professional noticing (Translated from Ivars et al., 2016, pp. 83-84)

5.2. Online discussions and interaction

In sociocultural approaches to learning, where knowledge is constructed through social interactions (Wells, 2002; Wenger, 1998), argumentation -as negotiation of meaning- plays a key role (Clark & Sampson, 2008; Llinares, 2012). Knowledge of a situation can be built through a process of dialogical argumentation that takes place when meanings are examined for reaching a consensus. For example, interactions between pre-service teachers in online debates, face-to-face interactions between in-service teachers and trainers, or written feedback between tutors and pre-service teachers (Shute, 2008) are manifestations of collaborative negotiation of meanings.

From the perspective of professional noticing of student mathematical thinking, interaction and feedback can help pre-service teachers to focus their attention on important mathematical elements of student thinking, recognize their understanding and make teaching decisions directed to student conceptual progress. In Fernández et al. (2012), participation in an online debate allowed students to reach a consensus on the mathematical elements identified in student responses, and interpret their understanding to propose teaching decisions. Pre-service teachers who had not considered those mathematical elements and, thus, had not identified profiles in student proportional reasoning (levels 1 and 2, Figure 15) began to identify features of student profiles (levels 3 and 4, Figure 15) after participation in the online discussion.

Table 3. Prospective teachers' level of expertise before and after on-line discussions

	Level of noticing before on-line discussion	Level of noticing after on-line discussion
Level 1	PT2, PT4, PT5, PT6	PT2
Level 2	PT7	
Level 3	PT1, PT3[a]	PT4, PT5, PT6, PT7[b]
Level 4		PT1, PT3[c]

[a] Both prospective teachers identify 2 out of 4 students' profiles
[b] Some of them identify 1 or 2 out of 4 students' profiles
[c] Both prospective teachers identify 3 out of 4 students' profiles

- *Level 1.* The prospective teachers do not discriminate proportional from additive situations. These prospective teachers only describe students' answers without relating the characteristics of the problem with the students' answers.
- *Level 2.* The prospective teachers discriminate proportional from additive problem relating students' answers with the characteristics of the problems, but they do not justify their answers attending to the mathematical elements of each situations.
- *Level 3.* The prospective teachers discriminate proportional from additive problem relating students' answers with the characteristics of the problems, and they justify their answers attending to the mathematical elements of each situations. However, they do not identify students' profiles.
- *Level 4.* The prospective teachers discriminate proportional from additive problem justifying trough the mathematical elements and identify the students' profiles.

Figure 15. Prospective teachers' level of expertise before and after online discussions (Fernández et al., 2012, p.753, 755)

Coles et al. (2013) showed that project meetings between in-service teachers, addressing low performance and creativity in class work, revealed changes in the way in-service teachers professionally noticed students' mathematical thinking. Thus, using the indicators proposed by Jacobs et al. (2010), changes were identified in the professional noticing of students' mathematical thinking. Finally, Ivars and Fernández (2018) showed that university tutor feedback on the narratives written by pre-service teachers during their practices period allowed them to focus on students' understanding according to the mathematical elements and led to teaching decisions (activities) directed to fostering students' conceptual progress.

5.3. Discussion and future perspectives

Our research converges with research by other international groups. Some studies have showed that the analysis of video clips illustrating interactions between teachers and students and between students solving problems (Santagata & Yeh, 2016; Schack et al., 2013; van Es & Sherin, 2002, 2008), or the interpretation of written students' responses focusing on errors (Cooper, 2009; Son, 2013) favour the development of professional competence. In this line, our research provides contexts that seem to facilitate this development, namely: writing of narratives during periods of practices at schools, online debates encouraging interactions between pre-service teachers or between pre-service teachers and trainers, face-to-face meetings, or written feedback.

Therefore, these results reveal which contexts can be used when designing teaching modules for pre-service teacher training programs to foster professional noticing of students' mathematical thinking. Other contexts such as sharing narratives with colleagues over online debates or writing narratives about their own practice represent future lines of research.

6. Final reflections

Our research is contributing to understanding which necessary characteristics teaching modules should possess when designing training programmes to develop the teaching competence of professional noticing of students' mathematical thinking: characteristics of the design of professional tasks (interpreting student responses with different degrees of understanding using a learning trajectory) and contexts that support their development (narratives and interaction).

Additionally, our research has begun to generate descriptors related to the degree of competence development in different mathematical domains, characterizing this development in relation to: the instrumentation of the hypothetical learning trajectory facilitated in the teaching module; progress in professional discourse to describe students' mathematical thinking; and the role of recognizing students' understanding of a concept's mathematical elements as a key to understanding student progression. These descriptors are making it possible to identify teacher learning trajectories in a similar way to the hypothetical learning trajectories of mathematical concepts applied to primary or secondary students (Simon, 2006). However, teachers' learning trajectories depend on how the following competence skills are articulated: identifying, interpreting and making decisions, and the relationships between them. Therefore, a future line of research for our group is to centre on designing teaching modules for pre-service primary and secondary school teachers based on empirically generated trajectories of competence development, as well as on the iteration of these cycles of design and review to improve teacher training practices.

To finish, most studies focusing on characterizing the relationship between professional noticing skills, degrees of competence development or examining contexts that favour competence development have centred mostly on pre-service primary school teachers. The fact that these studies were carried out to a lesser extent in secondary school is due to researchers' methodological challenges, i.e. lack of artefacts (videos or responses from secondary students) and missing theoretical references such as secondary students' hypothetical learning trajectories of certain mathematical concepts (Nickerson, Lamb & La Rochelle, 2017). Krupa, Huey, Lesseig, Casey and Monson (2017) highlight the need to explore the feasibility of tasks and contexts used at the elementary level, to develop competence in pre-service secondary school teachers. The structure of our designed teaching modules and professional tasks focusing on interpreting student responses to various problems with different levels of understanding, using a student learning trajectory designed ad hoc led this transfer to the secondary level.

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Noticing students' mathematical thinking: characterization, development and contexts

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In this paper, we summarise results from the research group of Didactics of Mathematics at University of Alicante (Spain) about professional noticing. The research focused on three issues over the last years: characterizing the relationship between the skills linked to professional noticing of student mathematical thinking, characterizing degrees of this competence development, and identifying contexts that support such development. The series of studies considered different mathematical domains (length magnitude and its measurement, fraction, generalization of patterns, proportionality, limit of a function at a given point, derivative of a function and classification of quadrilaterals) and educational levels (pre-service preschool teachers, pre-service primary and secondary school teachers). The three issues respectively address three questions. Firstly, how pre-service teachers notice student mathematical thinking (how they identify, interpret student mathematical thinking and decide according to their interpretations). Results led us to understand relationships between the skills of identifying, interpreting and making decisions, during initial training programs while teaching competence developed. Secondly, how pre-service teachers develop this competence. Results led us to generate descriptors of degrees of development. Our research has contributed to characterize pre-service teacher learning through: the instrumentation of the hypothetical learning trajectory given in the teaching module; the progression in professional discourse on student mathematical thinking; and the role of recognizing the understanding of certain mathematical elements of a concept as KDU. The descriptors are making it possible to identify teacher learning trajectories in line with the articulation of the skills of identifying, interpreting and making decisions, and the relationships between them. Thirdly, which contexts support the development of professional noticing. Our research concludes by suggesting some features of teaching modules that foster the development of teacher professional noticing in training programs, as well as the design of professional tasks and contexts. Major results in each issue are presented and discussed together with challenges for the future. A future line of research for our group is to centre on designing teaching modules for pre-service primary and secondary teachers based on empirically generated trajectories of competence development.