An approach to teach with variations: using typical problems

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Una aproximación para enseñar con variaciones: usando problemas típicos

Resumen

Los profesores de matemáticas usan problemas típicos desde propuestas de exámenes anteriores y desde los libros de texto para desarrollar destrezas procedimentales. En este articulo, discutimos otros usos de los problemas típicos. Nos centramos en las oportunidades que un profesor experimentado, John, percibe en los problemas típicos y cómo los usa para potenciar el aprendizaje de sus estudiantes aprovechando las variaciones del problema(o bianshi). A partir de los datos de una investigación con enfoque cualitativo centrada en la competencia "mirar profesionalmente" del profesor, presentamos una instantánea de la práctica de John para mostrar lo que observa de las posibles variaciones en los problemas típicos y cómo las usa con los estudiantes para promover tanto las destrezas procedimentales como la comprensión conceptual. Los resultados subrayan el potencial de apoyar a los profesores para que aprovechen las variaciones de los problemas típicos, lo cual tienen implicaciones para la formación inicial y continua de profesores.

Palabras clave. Tareas matemáticas; formación de profesores; enseñar con variaciones; problemas típicos.

Uma abordagem para ensinar com variações: usando problemas típicos

Resumo

Os professores de matemática usam problemas típicos de propostas de exames anteriores e de livros didáticos para desenvolver procedimentos. Neste artigo, discutimos outros usos de problemas típicos. Nós nos concentramos nas oportunidades que um professor experiente, John, percebe nos problemas típicos e como ele os usa para melhorar a aprendizagem de seus alunos aproveitando as variações do problema (ou bianshi). A partir dos dados de uma pesquisa qualitativa voltada para a competência "olhar profissionalmente" do professor, *apresentamos um instantâneo da prática de John para mostrar o que ele observa de possíveis variações em problemas típicos e como ele os usa com os alunos para promover habilidades processuais e compreensão conceitual. Os resultados destacam o potencial de apoiar os professores para tirar proveito das variações de problemas típicos, que têm implicações para a formação inicial e contínua dos professores.*

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Palavras chave. Tarefas matemáticas; educação de professores; ensinar com variações; problemas típicos.

An approach to teach with variations: using typical problems

Abstract

Mathematics teachers use typical problems from past examination papers and textbook exercises to develop procedural skills. In this paper, we discuss other uses of typical problems. We focus on the affordances that an experienced teacher, John, perceives in typical problems and how he uses them to enhance student learning by harnessing the idea of teaching with variations or bianshi. Drawing on data from a larger qualitative design-based research on investigating teacher noticing, we present snapshots of John's classroom practices to show what he noticed about the variations afforded by typical problems and how he used these problems with students to promote both procedural skills and conceptual understanding. Findings suggest the value of supporting teachers in harnessing variations of typical problems, which has implications for teacher education and professional development.

Keywords: Mathematical tasks; teacher education; teaching with variations; typical problems.

Une approche pour enseigner avec des variations: utilisation des problèmes typiques

Résumé

Les professeurs de mathématiques utilisent des problèmes typiques des propositions d'examen précédentes et des manuels pour développer des compétences procédurales. Dans cet article, nous discutons d'autres utilisations de problèmes typiques. Nous nous concentrons sur les opportunités qu'un enseignant expérimenté, John, perçoit dans les problèmes typiques et comment il les utilise pour améliorer l'apprentissage de ses étudiants en profitant des variations du problème (ou bianshi). À partir des données d'une recherche qualitative axée sur la compétence «noticing» de l'enseignant, nous présentons un aperçu de la pratique de John pour montrer ce qu'il observe des variations possibles dans les problèmes typiques et comment il les utilise avec les élèves promouvoir les compétences procédurales et la compréhension conceptuelle. Les résultats soulignent la possibilité d'aider les enseignants à tirer parti des variations de problèmes typiques, qui ont des implications pour la formation initiale et continue des enseignants.

Paroles clés. Tâches mathématiques; formation des enseignants; enseigner avec des variations; problèmes typiques

1. Mathematical tasks and typical problems

Mathematics teachers invariably orchestrate lessons using a multitude of tasks. Traditionally, teachers have used typical problems such as examination-type questions and standard textbook exercises, to develop procedural skills. Despite the "omnipresence" of typical problems in the mathematics classroom, research into their use for developing conceptual understanding is limited. Do typical problems have affordances for developing conceptual understanding? In this study, we focus on the following research questions: (1) What affordances do teachers perceive in typical problems, and (2) How do they use typical problems in the classroom to enhance student learning? We describe how an experienced teacher, John, used typical problems to develop both procedural skills and conceptual understanding by harnessing the idea of teaching with variations or *bianshi*.

Mathematical tasks are central to student learning because they convey meaning about what mathematics is and what doing mathematics entails. The answer to what constitutes a task in the mathematics classroom may depend on the perspectives of both the teacher and the students. The teacher sets tasks for the students to work on and elicit particular learning outcomes. Stein, Grover and Henningsen (1996) encapsulate this view in their description of task as "classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (p. 460). Along the same lines, Watson and Thompson (2015) refer to the written presentation of a planned mathematical experience for a learner, which could be one action or a sequence of actions that form an overall experience. Thus, a task could consist of anything from a single problem, or a textbook exercise, to a complex interdisciplinary exploration. Questions set by teachers for groups of students to work on are considered as tasks too.

Mathematics teachers use a variety of tasks during lessons to develop mathematical competencies in students. Efforts on orchestrating productive mathematical discussions (Smith & Stein, 2011) have amongst others been focused on the use of rich mathematical tasks (Grootenboer, 2009), challenging tasks (Sullivan et al., 2014), high-level tasks (Henningsen & Stein, 1997) and open-ended tasks (Zaslavsky, 1995). Although these tasks offer opportunities for students to do mathematics (Smith & Stein, 2011), they also present obstacles in implementation. First, these tasks may have too high an entry point for many students so that teachers have to provide additional prompts or scaffolds (Sullivan et al., 2014). Next, it takes time and effort for teachers to select, adapt or design challenging tasks for use in the classroom. Third, the inherent complexity of the rich tasks involves mathematics from across the curriculum, and results in these tasks implemented across several lessons. Consequently, these obstacles place high demands on teachers' knowledge and time, and may limit the incidence of such tasks in the mathematics lessons.

Furthermore, teachers may face challenges during the implementation of a lesson based on a single mathematically-rich task (Grootenboer, 2009) because students may not engage with it as intended. The centrality of a task is restrictive as it constrains the teacher and the flow of the lesson. It may not be easy to bring in a new task to re-engage students when they lose interest, or digress during the lesson to address questions and misconceptions. Teachers are pressed by the concurrent need to focus on honing procedural fluency as part of standardised testing preparations. This may lead to the implementation of a high cognitive-demand task as a low cognitive-demand task (Stein et al., 1996). These concerns are particularly true in an examination-oriented country such as Singapore, where the use of textbook- and examination-type questions are common.

As in many countries, completing the syllabus and preparing students for examinations are genuine concerns of teachers in Singapore. It is thus common for teachers in Singapore to adopt a teacher-centred teaching approach and use examination-type questions to develop procedural skills (Ho & Hedberg, 2005). This preference for using typical problems—standard examination or textbook problems—reflect teachers' belief that it is more "important to prepare students to do well in tests than to implement problem-solving lessons" (Foong, 2009, p. 279). We cannot ignore this reality. Despite the widespread use of typical problems in mathematics classrooms to develop procedural skills, research into their use for developing conceptual understanding remains limited.

In Choy and Dindyal (2017a, 2017b), we describe how an experienced teacher, Alice, used typical problems to develop relational understanding through procedural skills and conceptual fluency. Using Gibson's (1986) ideas about affordances, we emphasise that: (1) an affordance for using a typical problem exists relative to the action and capabilities of the teacher, (2) the existence of the affordance is independent of the ability to perceive it, and (3) the affordance does not change as the needs and goals of the teacher change. Following Gibson, affordances in relation to an observer can be positive or negative which in our context may lead to more or less productive use of the problems in class. Hence, to perceive the affordances of a typical problem implies noticing the characteristics of the task in relation to the understanding of the related concepts and to its adaption for use. That includes noticing the mathematical connections encapsulated in the task and orchestrating discussions around connections to enhance learning.

2. Noticing mathematical connections in tasks

As Smith and Stein (2011) highlight, the quality of a mathematics task is critical for developing mathematical proficiencies through classroom discussions. Selecting high-quality tasks requires teachers to focus their attention on the mathematical elements embedded in the tasks. For example, teachers may want to attend to the cognitive demand in terms of the mathematical processes required to solve the task (Smith & Stein, 1998). Besides the cognitive demands of a task, it may be useful for teachers to design tasks around students' possible confusion about the concepts they are learning (Choy, 2016). Whether teachers select, modify or create tasks for use in class, they have to see and make sense of the mathematics and pedagogical considerations in the tasks. This specialised seeing, sense making, and decision making is a set of three inter-related skills referred to by researchers as teacher noticing (Mason, 2002; Sherin, Jacobs & Philipp, 2011).

The professional vision called noticing can be viewed as a set of practices that work together to improve teachers' sensitivity to act differently in teaching. Mason (2002, p. 61) distinguishes disciplined from spontaneous noticing by indicating its systematic aspect:

> The idea is simply to work on becoming more sensitive to notice opportunities in the moment; to be methodical without being mechanical. This is the difference between 'finding opportunities' and 'making them'. Instead of being caught up in moment by moment flow of events according to habits and pre-established patterns, the idea is to have the opportunity to respond freshly and creatively yet appropriately, every so often.

There are two main ways, as Mason (2002) puts it, to raise the possibility of noticing in order to respond freshly or have a different act in mind for the future: advance preparation and learning from experience. In the case of using typical problems to develop relational understanding, teachers will need to notice the potential use or affordances offered by the tasks beyond developing procedural skills. We see noticing as productive when teachers perceive and harness the affordances of typical problems to develop procedural skills and conceptual understanding. An issue here is what makes noticing productive. Choy, Thomas and Yoon (2017) characterise *productive noticing* in terms of having an explicit focus for noticing through pedagogical reasoning—i.e. how teachers justify their instructional decisions or claims about student thinking using specific and appropriate details of what they have attended to. This notion of productive noticing builds on Yang and Rick's (2012) Three Point Framework by suggesting that an explicit focus is useful for supporting teachers to notice relevant instructional details during the planning, teaching and reflection of lessons (Choy et al., 2017). There are two key aspects of this focus. First, the three components of the didactical triangle, namely the mathematical concept, students' confusion associated with the concept, and teachers' course of action to address such confusion. Second, the alignment between these three components, that is, whether the teacher's course of action targets students' confusion when they are learning the concept. Ensuring this alignment between teachers' instructional decisions and students' confusion is not trivial, and it is mediated by teacher's pedagogical reasoning (Loughran, Keast & Cooper, 2016; Sánchez & Llinares, 2003; Shulman, 1987). Extending this notion of productive noticing, we see teachers' noticing as productive when he or she is able to see the connections between the problem and the curriculum in terms of what to teach—concepts, conventions, results, techniques, and processes (Backhouse, Haggarty, Pirie & Stratton, 1992). In addition, the teacher needs to make sense of how the problem can feature in a sequence of other typical problems, and use these typical problems to develop relational understanding (Choy & Dindyal, 2017a, 2017b).

3. Orchestrating discussions

Stein, Grover and Henningsen (1996) have provided a model for instructional tasks used by teachers to elicit desired learning outcomes in students while focusing on a particular concept, idea or skill. This model also applies to typical problems thought of as mathematical tasks: (1) represented in curricular or instructional materials, (2) set up by the teacher in the classroom, and (3) interpreted by students in the classroom (Figure 1).

Although the model provides a way to think about the representation, setup and implementation of tasks, the features of a task are necessary but not sufficient for enhancing student reasoning (Henningsen & Stein, 1997). Based on the analysis of 58 tasks (out of 144) that might afford 'doing mathematics' (Smith & Stein, 1998), it was found to be critical for teachers to support student reasoning by "pressing" them to "provide meaningful explanations or make meaningful connections", without "reducing the complexity and cognitive demands of the tasks" (Stein et al., 1996, p. 546). This is what Mason and Johnston-Wilder (2006) termed as "scaffolding and fading" (p. 83). Therefore, how a teacher engages students with the task during task implementation is of utmost importance in supporting their mathematical reasoning.

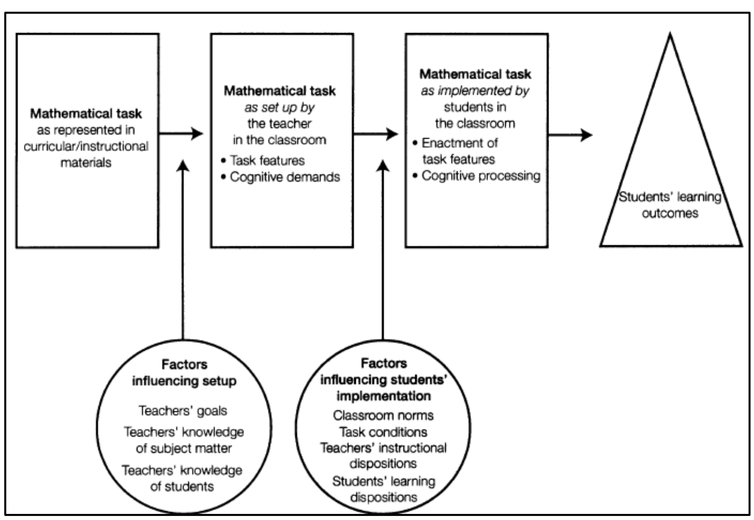

Figure 1. Mathematical tasks (Stein et al., 1996, p. 528)

The use of classroom discussions to engage students with the task is not trivial. There is a need for teachers to give students time to explore and work on the tasks; on the other hand, it is crucial for teachers to use students' responses to the tasks, and build on them to advance mathematical understanding. According to Stein, Engle, Smith and Hughes (2008), teachers can use students' correct, partially correct, and incorrect responses to tasks as initiators of discussion. The crux is to facilitate classroom interaction for shaping student mathematical reasoning, which is the hallmark of a well-orchestrated discussion. This study presupposes that orchestrating discussions is "deliberate work" (Franke, Kazemi & Battey, 2007, p. 228), and certain aspects of this teaching expertise can be planned. Stein et al. (2008, p. 321) introduced five practices:

- *anticipating* possible student responses to a task;
- *monitoring* their responses when students work with the task;
- *selecting* students purposefully to present their work;
- *sequencing* their presentations carefully to build up mathematical ideas; and
- *connecting* students' responses to each other and to the underlying mathematical concepts

Smith and Stein (2011) highlight the importance of using a mathematically rich task for orchestrating mathematically productive discussions. They suggest an instructional sequence that centres about a single rich task in which students attempt, present, and discuss the mathematics under the orchestration of a competent mathematics teacher. However, in Choy and Dindyal (2017b) we suggest an alternative. Our teacher participant, Alice, orchestrated mathematics discussions through a careful sequencing of simpler tasks involving typical problems. The structure of Alice's lesson differs from that envisioned by Smith and Stein (2011) in the plurality of tasks within the same lesson, punctuated by several more rapid successions of the same discussion moves: monitoring, selecting, sequencing, and connecting. This structure is made feasible by the use of typical problems, which generally take a shorter time to complete. In the rest of the paper, we focus on how typical problems can be set up differently for orchestrating discussions in the classroom to develop relational understanding by using the idea of variation.

4. Harnessing variation

The variation theory proposed by Marton (2014) uses a phenomenographic approach. The key idea is that learners will notice what is varying against a background of invariance. If too many things vary then individual variation is obscured (Watson & Mason, 2005). The implication for teachers is that mathematics tasks should be designed so that the desired content (known as the critical aspect in the theory) is varied and learners can see this and the effects of such variation in successive examples (Watson & Thompson, 2015). This critical idea has been termed the object of learning by Marton and Pang (2006). The object of learning includes the direct object of learning (content) and the indirect object of learning (capability of using that content). The teacher's role is then to create a pattern of variation and invariance, with the object of learning in mind, which the student must experience to learn. This pattern of variation and invariance is distinguishable more readily in a typical problem than in the so-called 'rich tasks' (Grootenboer, 2009).

The pattern of variation and invariance is better discernable in typical problems possibly because these problems offer more opportunities for “repetition by systematically introducing variations” (Wong, 2008, p. 977). As argued by Wong, Lam, Sun and Chan (2009), this form of repetitive learning, common in China and other East Asian countries, differs from rote learning. The difference lies in teaching with variation, practised in the form of *bianshi* (变式) teaching in China since the 1980s. While the pedagogy of variation emphasises concept development, *bianshi* teaching enhances problem-solving (Gu, Huang & Marton, 2004). We view *bianshi* as a form of variation, and follow the distinction made by Gu et al. (2004) between conceptual and procedural *bianshi*. Building on this notion, there are four basic types of *bianshi*: inductive, broadening, deepening, and applying (Wong, Lam & Chan, 2013).

In inductive *bianshi*, teachers use a series of carefully selected examples or situations for students to discern the critical features of a concept or skill. Teachers may use real-life examples of housing loans, hire purchases, and investments to highlight the idea of compound interest. Students develop the basic formula of compound interest accrued annually by examining each example. Following this, students consolidate their understanding by experiencing variations, or broadening *bianshi* introduced into the tasks by teachers. The aim of these tasks is not to introduce new concepts, but to see instances or uses of the concept to be learned. Examples of such tasks may include questions involving annual compounding of interest in a variety of contexts. In contrast, deepening *bianshi* aims to expand student understanding by varying certain aspects of the mathematical concept or skill. For instance, students can deepen their understanding of the compound interest formula when they are exposed to questions in which the interest rates are not annual interest rates, and the compounding frequencies are no longer annually. Finally, teachers use applying *bianshi* to promote students’ application of their new understanding to solve a variety of more realistic problems involving the notion of compounding.

We focus on variations as a lens to investigate what teachers notice about mathematics in typical problems. We present the case of John (pseudonym), an experienced teacher, to highlight the use of these problems to develop relational understanding. We describe how John sequenced a set of similar typical problems, harnessing on slight variations between the problems in his move from teaching for instrumental understanding to teaching for relational understanding.

5. Context for the case of John

This study draws on data collected from a larger design-based research (Design-Based Research Collective, 2003) aimed at developing a toolkit to support teachers in noticing relevant instructional details, and refining a theory to describe their noticing when orchestrating learning experiences. We went through three iterative cycles of theory-driven design, classroom-based field testing and data-driven revision of the Mathematical Learning Experience Toolkit (MATHLET) to provide a theoretical justification for the underlying analytical frameworks. Four experienced mathematics teachers from three secondary schools, with different achievement bands and demographic factors, participated in this project. In each design cycle, the teacher participants designed and implemented a lesson of their choice using the MATHLET. This resulted into 12 design cycles across three schools. Data consisted of voice recordings of planning, pre-lesson and post-lesson discussions, video recordings of lessons and lesson artifacts. By collaborating with teachers in designing, implementing and reviewing learning experiences using the MATHLET, we aimed to develop a deeper understanding of how teachers orchestrate mathematically meaningful learning experiences in different classroom contexts. Findings were then developed using a "thematic approach" (Bryman, 2012, p. 578) together with the two characteristics of productive noticing as proposed by the FOCUS Framework (Choy et al., 2017). John is one teacher of the project who used typical problems in his teaching with variations.

John is an experienced mathematics teacher who has been teaching high-achieving students for more than 20 years. He has a strong subject mastery with a Master degree in Mathematics and an Honours degree in Computer Science. John is proficient in the use of technology for teaching, especially graphing calculators. As a Senior Teacher in his school, John has also demonstrated pedagogical content knowledge, and actively engaged his colleagues in professional development. In many ways, John's beliefs about mathematics teaching and learning reflect that of a connectionist teacher in numeracy (Askew, Rhodes, Brown, William & Johnson, 1997). He wants his students to be aware of the diversity of methods and know when to use them. For example, during our interview with him, he mentioned how he had focused on creating learning experiences for students where they had opportunities to reason about the most appropriate method:

> A simple thing like prove that ABCD is a parallelogram. I can do it using geometry, just draw it out and show you that it is a parallelogram. I can do it using vectors; I can using coordinate geometry. Same question, three different approaches. Which one do you choose? Now, that is the- so therefore the students thinking processes, how do I choose the correct, or the most applicable, method to answer that question and then answer it?

In addition, John sees the mathematics curriculum as a connected whole and makes connections with other topics. In his teaching, he tries to highlight how a single question can be solved using different approaches, drawing on connections between topics. Thirdly, he emphasises on understanding the concept underpinning the procedures. In one post-lesson discussion, he termed his notion of concept underpinning the procedures as a "procedural concept":

> So we always wonder, let's say for example I give a question, let's say for example um. . . um, x squared minus 5x plus 6 equals to zero. Our teachers always tell our kids not to bring things over, keep right-hand side to zero. Why? Why do I keep the right-hand side to zero?

John's idea of a procedural concept can be seen from his emphasis on understanding the reasons behind the procedure of "keeping the right-hand side to zero", which is usually not expected at that level of study. His thinking reflects a connectionist belief aligned with his goal to teach the key ideas of mathematics—to highlight connections between mathematics and real life problems, and between mathematical topics within and beyond the secondary school levels.

The vignettes, developed from video and voice recordings of two lessons observed at a Secondary Three (Grade 9) classroom in Spring Hill School (pseudonym), are illustrations of what John typically does as he comes to the end of a topic. The first vignette focuses on a lesson on compound interest, where John tried to convey the meaning of the variables in

$$P = \left(1+\frac{r}{100}\right)^n$$

P refers to the principal sum, *r* the interest rate for a given compounding frequency, and *n* the number of times interest is compounded. He tried to direct students' attention to situations with use of different compounding frequencies and rates. The second vignette highlights a revision lesson on solving trigonometric equations through a sequence of four illustrative examples. In this lesson, John drew on what he noticed about students' mistakes and designed four trigonometric equations for students to work on:

$$3\sin\theta + 4\cos\theta = 0,\ 3\sin\theta + 4\cos\theta = 1,\ 3\sin^2\theta + 4\cos\theta = 1,\ 3\sin 2\theta + 4\cos\theta = 0$$

The two lessons took place almost six months apart, but John's approach of harnessing variations was similar.

6. Two vignettes of harnessing variations

Compound Interest Vignette

John had taught how to find simple interest and compound interest. He recalled the formulae for a quick review and proceeded to deepen the understanding of the compound interest formula.

John: … You invest $10,000 in an account that gives 3% per annum, compounded annually. Before you do anything, ask yourself one question. What is the keyword you must look for here? [*writes the duration to be 5 years*]

S1 : Compounded

John: Correct. Compounded annually. That's the first keyword. Give me the second keyword.

S2 : Per annum.

John: Sorry? Correct, per annum. Why is this important? Sorry?

S3 : Compounding once per year.

John: Correct, excellent. You must make sure that the annual compounding and the rate is the same duration. You must be very careful. We'll see why in a little while. Let's now work this out. So, what's your answer? Your total will be... Give me the formula, I don't want the answer, you know the answer, I want the formula.

S4 : [*uses the appropriate numbers in the compound interest formula and reads it for John*]

John: 10,000 times 1 plus 3 over 100... very good. Answer please? 11592.74. Exactly correct to 2 dp. Very good. Now, let's change the question. What if I told you, I don't want to compound annually. I want to compound quarterly.

S5 : Divide the rate into 4.

John: Aha! So! Very good, this is 3% per annum compounded annually. Now, it's per annum compounded quarterly. So what happens? Your time is now shortened by a factor of 4, which means that the rate must also be factor of 4. Very good. So your total will be 10,000, 1 plus...times? Sorry? What's wrong? Sorry?

S6 : You're calculating in quarters.

John: Because, yeah correct, because you're compounding quarterly right, so there are how many quarters in 5 years? Because remember? The rate was per quarter, this is quarter, this is quarter, right? Therefore you must make sure that the 2 rates have the same timeframe. So answer please, someone? 116114 to 2 dp. Very good. Now, one last question. Can you now tell me - ok, what if I told you this? 3% per annum compounded quarterly, quarter. I want 3% per quarter compounded quarterly. So what's the formula?

As seen from his exchange with the students, John made a series of subtle, yet critical changes to the interest rate problems to highlight how students should make adjustments to the formula for compound interest. He started by varying the compounding frequency from annual to quarterly to bring forth the necessary changes to the formula, before moving to other changes involving the interest rates. Here, we see how John used the idea of *bianshi* to support students in understanding the different components of the formula.

Trigonometric Equation Vignette

John wrote four trigonometric equations on the board (Figure 2) and asked the students to discern the differences:

> What I've noticed yesterday was as you were doing the work, you know how to start, but when you get to a point you got confused because you don't know how to continue. Now let's go uncover these 4 questions. Now, take the first 2 minutes in each group, 3 of you, tell me what are the main differences in all 4 questions. Don't give me cosmetic differences. Oh, this got plus got minus got this...don't need. I don't want cosmetic, I want theoretical, conceptual differences, ok? So ask yourself 2 questions. Number 1, when you do trigonometry, what are the 2 most important things to remember? Number 2, when you do trigonometry, what are the other considerations that you must account for when you solve a trigonometric equation. Those are the first 2 questions. 1 minute, talk, buzz, go.

Figure 2 shows that the four equations look similar but are structurally different. The difference between the first ($3\sin\theta+4\cos\theta=0$) and the second ($3\sin\theta+4\cos\theta=1$) lies in the number on the right-hand side. This variation changes the structure and solution method. In the first equation, students divide both sides by $\cos\theta$ to obtain an equation containing only the tangent function. The second equation requires them to transform the equation into $R\sin(\theta+\alpha)=1$.

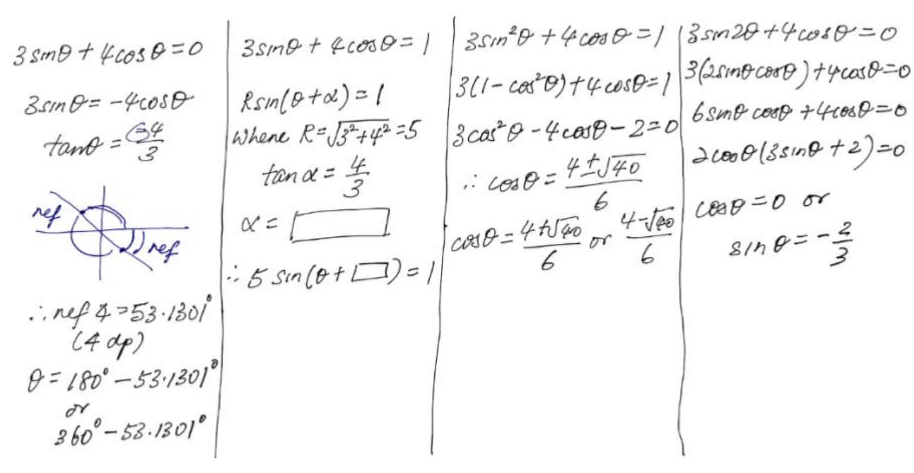

Figure 2. John's selection of the four trigonometric equations

John wanted his students to pay attention to the change in structure and solution method. He opens up opportunities to see the difference for themselves, and to understand why they cannot divide both sides by $\cos\theta$ for the second equation:

John:	Finished? Ok, method, answer?
SA:	The first one is tangent.
John:	Wait, no, no, yeah but what makes it different? What makes the difference?
SB:	Because it's value of...
John:	Sorry? Yeah so?
SC:	Yeah then there's sine and cosine, so you can become tangent.
John:	So why can't I do that for the second one?
SD:	Second one is 1.
John:	So? So why cannot, tell me, why can't you do it?
SE:	Huh, because the value will jump.
John:	So? What's the problem?
SF:	When you manipulate you cannot do the division...
John:	Can... why you divide by 1? You divide by cos right? So why can't do 3 tan θ plus 4 equals um...
SG:	Because that will give you 2 different...
John:	So why can't I do that here?
SH:	Because there's the 1.
John:	But can become [secant] mah. You mean can't become [secant] ah? You mean you cannot solve this? You mean this one I cannot solve? Then? Then? What's wrong with this? no no I understand your reason, I'm not saying you're wrong, but my question is, you say I cannot divide by cos θ, right? But I can, you can solve this, because this will simply be... So it's not impossible. So, but why is it - it's not easy this time? …
SI:	1 tangent.

John's use of the four equations is an example of *bianshi*, which involves both conceptual and procedural elements. For instance, John tried to get students to see through the surface structural differences (i.e., 0 versus 1 on the right-hand side) and understand the key to solving trigonometric equations—to reduce the equation into one with a single trigonometric function (conceptual). In the third and fourth equations, John varied the equations by introducing a squared term in the third and a change in angle in the fourth, while maintaining some similarities with the previous equations. Although this change in the structure of the equation requires students to use a different solution method (procedural), John wanted them to see that the key idea of solving trigonometric equations remains the same (conceptual). This excerpt is typical of how John orchestrates the mathematical discussions during his lessons. His practice of conducting discussions reflects a connectionist's belief in that he uses "focused" discussions to "help pupils explore efficient strategies and interpret the meaning of mathematical problems" (Askew et al., 1997, p. 32).

7. Discussion and conclusion

The two vignettes above show how John harnessed the idea of teaching with variations or *bianshi* to develop relational understanding. In the Compound Interest Vignette, John used a rapid succession of scenarios, which involves compounding frequencies and the kind of interest rate given, to deepen student understanding. Building on the standard formula from textbooks for compound interest, he used a sequence of short questions involving these variations to highlight the corresponding changes to the formula. This is different from what many would term as “rote learning” because of his emphasis on the reasons behind differences in the procedures. His choice of questions and the sequencing were deliberate to reflect the key idea behind the formula. He used his assessment of student understanding to design a series of questions:

> Um, right now what I do is, because I know what concepts, what procedural concepts I'm testing today, I can then analyze um, so the question I will set will be based on that one skill. So every assessment I give them, every question I set them will have a specific skill that I'm testing. Either a specific concept, or a specific algebraic manipulation skill that I'm testing. Every question will have something that I'm looking for.

By seeing how the compound interest formula varies given the *r* and the compounding frequency, John used short questions to highlight the key point he was trying to teach. His use of typical problems with slight variations aimed at deepening student understanding of the formula beyond developing procedural skills. This is an example of his use of typical problems by harnessing deepening *bianshi.*

In the Trigonometric Equation Vignette, John used a series of four typical problems to highlight differences in the structures of the four equations, and guided students to notice the strategies in solving trigonometric equations. He used four similar questions that vary in the structure, while keeping the coefficients of $\sin\theta$ and $\cos\theta$ constant. He deliberately designed the four equations to illustrate the four basic types of trigonometric equations common in examinations. However, John went beyond preparing students for examinations by highlighting the thinking processes required to solve trigonometric equations. The design of the equations support students to make sense of the structural differences in the equations and connect these differences to the corresponding solution methods. This is especially so for the first equations, which look similar but their solution methods are different. John wanted to broaden student understanding of the solution methods by raising awareness of structural differences.

As seen in the two vignettes, John noticed his students’ errors and was cognisant of the key idea for the lesson. For example, in the Compound Interest Vignette, John used his insights into students’ understanding and possible confusion, and designed the sequence of short questions to denote what is invariable in the questions (“the two rates have the same timeframe”). Similarly, in the Trigonometric Equation Vignette, he designed the four questions around what students had previously found to be challenging. During the post-lesson interview after the review lesson, John highlighted his thinking and reasons for designing the four questions:

> I know that for that class, because I know where their problems are, I know which problems will cause them problems. So if I give them a simple question, for example, $3\sin\theta$ equals to $4\cos\theta$, oh they can solve that no problem, they can definitely do that without an issue. But the minute I change something else or I add a constant to it, if I add something else to the whole thing, if I change the double angle for example, they find that part very very uncertain.

Besides noticing the specificities of content and confusion when learning the topics, John perceived the affordances of typical problems and considered how he could harnessed the types of variation to address errors. John was able to reason about his modification or choice of typical problems. Hence, John's noticing is productive as he harnessed the idea of variations or *bianshi* in his use of typical problems to teach for relational understanding (Choy et al., 2017).

In terms of orchestrating discussions, John used a series of short, focused questions to engage his students. While John's questioning may be similar to the classic Initiate-Respond-Evaluate pattern (Greeno, 2003), he listened to students' responses and guided them based on what they were thinking. As a result, his questioning is more focusing rather than funnelling (Wood, 1998). He achieved this focusing through the deliberate sequencing of the problems modified. He listened to students and responded to further their relational understanding. This is more evidence for considering John's noticing productive (Choy et al., 2017).

John harnessed the idea of variations or *bianshi* by making deliberate modifications to typical problems for broadening and deepening student understanding of the skills. He tried to guide students in making connections between the procedural skills and the concepts they had learned. His use of typical problems was characterised by deliberate changes to the structure of the chosen problems to highlight specific aspects of the concept or skill. This stands in contrast to Alice, as described in our earlier work (Choy & Dindyal, 2017b). In the case of Alice, she modified the typical problems to open up the solution space, which provided opportunities for students to use different methods to solve the problem. Alice used students' responses to the typical problems to develop relational understanding by connecting their responses to key mathematical ideas in the same topic. Hence, we see two different approaches to using typical problems for developing both procedural skills and conceptual understanding.

In his interview, John highlighted how he unpacked the curriculum by thinking more deeply about what students were supposed to learn beyond the skills. He paid attention to the specific concepts, conventions, results, techniques, and processes in a given topic (Backhouse et al., 1992). Similar to Alice, John attended to the structure of a unit by thinking of it as a sequence of lessons, which comprised of a sequence of tasks, and considered how he could encapsulate the mathematics in the tasks. He was aware of the connections between the concepts within and beyond the topic. While acknowledging that he is an experienced teacher, we believe that other teachers can develop such professional vision—productive noticing of the curriculum. To this end, we propose three approaches for future research:

Beyond learning new content. Rather than learning new content, there is an issue with using teachers' knowledge to delve deeper into school mathematics. It is more about supporting teachers to use what they know, and guiding them to see new connections between aspects of the mathematics they are teaching. It is about guiding them to see the forest and the trees. Teachers need to have opportunities to zoom in and out of the curriculum, and notice systematically its details (Mason, 2011). In particular, they have to learn how to attend to the whole curriculum; discern the details of the concept; seeing the teaching of this concept in a sequence of lessons; conceptualising a lesson as a sequence of tasks, and encapsulating the mathematics within the tasks, paying attention to inter typical problem differences.

Typical problems. The omnipresence of typical problems offer opportunities for teachers to enhance student learning experiences in a more pervasive manner. We see one area of potential professional development in supporting teachers to notice the affordances of typical problems. There are at least two ways to think about the affordances of typical problems. First, as in Alice's way of modifying problems to expand solution space. This approach provides multiple entry points for different groups of students. Next, as described in this paper, John modified the problem to restrict the solution space to specific cases. This "zooming in" allows teachers to highlight the critical features of the concept. Both ways to think about typical problems are critical if we want to use them for developing relational understanding.

Conversations about students thinking. Many professional development approaches have centred on having professional conversations about teaching and learning. However, not all conversations are productive for enhancing student thinking. As argued by Lee and Choy (2017), it is crucial for teachers to focus on specificities of the concept, what students find difficult when learning the concept, as well as teachers' approaches to address these learning challenges. Choy et al. (2017) highlight the importance to notice the alignment between the triad of teaching and learning—content, student thinking, and teacher actions. Focused conversations should be supported in order to harness the potential of typical problems for enhancing student learning.

Although the examples used in this paper came from a single teacher, we have seen similar approaches from other teachers in our studies. While we acknowledge that some teachers use typical problems in a limited fashion, we see potential in exploring and enhancing the use of such problems to develop conceptual understanding. The idea of teaching with variations (Gu et al., 2004) provides one avenue to explore this fertile terrain.

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An approach to teach with variations: Using typical problems

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Mathematics teachers use typical problems from past examination papers and textbook exercises to develop procedural skills. In this paper, we discuss other uses of typical problems through the following research questions: (1) What affordances do teachers perceive in typical problems, and (2) How do they use typical problems in the classroom to enhance student learning? We focus on the affordances that an experienced teacher, John, perceives in typical problems and how he uses them to enhance student learning by harnessing the idea of teaching with variations or bianshi. Drawing on data from a larger qualitative design-based research on investigating teacher noticing, we present snapshots of John's classroom practices to show what he noticed about the variations afforded by typical problems and how he used these problems with students to promote both procedural skills and conceptual understanding. Findings suggest the value of supporting teachers in harnessing variations of typical problems, which has implications for teacher education and professional development. While we acknowledge that some teachers use typical problems in a limited fashion, we see potential in exploring and enhancing the use

of such problems to develop conceptual understanding. We importantly acknowledge the development of future research regarding the need to: i) Support teachers to use what they already know by means of establishing newer connections rather than learning further content. ii) Enhance opportunities for teachers to enhance student mathematics learning based on the affordances of typical problems. iii) Develop contexts for professional conversation about student mathematical thinking that focus on specificities of the concepts of teaching and learning.