

# The mathematics that secondary teachers (need to) know

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In spite of decades of focused research, the phenomenon of «teachers' disciplinary knowledge of mathematics» is not yet a well-formulated construct. This lack of deep understanding is troublesome, as uncritical assumptions about and entrenched practices around the mathematics that teachers should know can have major implications for teacher preparation programs, especially at the secondary level.

For example, in the vast majority of Canadian universities with programs in secondary mathematics teacher education, candidates are required to complete a suite of stock mathematics courses —even though there is little evidence that meaningful connections exist between these courses and teachers' effectiveness in the classroom. The inability to identify such relationships has been a source of frustration among researchers for some time (cf. Begle, 1972, 1979). Yet, in part because the field has not yet offered consistent and substantial evidence around alternative constructs, most teacher education pro-

grams have evolved little —in North America at least.

Indeed, even among researchers, the lack of evidence around this matter has done little to diminish the conviction that university-level courses in mathematics are vital to teacher preparation. Consider, for example a conclusion drawn by Baumert and colleagues (2010) in a comprehensive review of empirical research on the issue:

Findings show that [teachers' content knowledge of mathematics] remains inert in the classroom unless accompanied by a rich repertoire of mathematical knowledge and skills relating directly to the curriculum, instruction, and student learning. ... In summary, findings suggest that —in mathematics at least— a profound understanding of the subject matter taught is a necessary, but far from sufficient, precondition for providing insightful instruction. (p. 139)

The usage of the word *inert* is notable here. It can be tempting to read it as a hedging term —that is, as a tactic to sidestep the unresolved issue of how stock courses in mathematics matter to teaching. Such a move would enable the field to maintain the unsubstantiated conviction that formal courses do matter, in spite of a lack of evidence.

Whether or not that might be the case, a different advantage to thinking in terms of «inert» rather than «not demonstrated as useful» knowledge is that it focuses attentions in teacher preparation away from efforts to measure the impact of background disciplinary knowledge and toward efforts to activate that knowledge. That is, the notion of «inert mathematics knowledge» signals several shifts in thinking around mathematics teachers' disciplinary knowledge. For one, it flags a new focus in the research, whereby the longstanding concern with «what» mathematics teachers should know has been extended to encompass «how» they need to know it. It also hints at a popular current idea that teachers' disciplinary knowledge is perhaps better understood in terms of many and varied facets rather than a consolidated monolith.

Unfortunately, even with a shift toward thinking in terms of «activating inert knowledge» rather than «adding more stock knowledge» —coupled to the fact that there is broad agreement that teachers' disciplinary knowledge comprises diverse elements— there is no consensus on which knowledge and skills are most critical for bringing teachers' disciplinary knowledge to life in the classroom. Two perspectives are prominent, but neither

has a research base that enables strong claims about practice. The majority of current studies focus on explicit knowledge of curriculum content and instructional strategies. Such knowledge might be assessed directly through observation, interview, or written test (e.g., Ball, Hill, & Bass, 2005; Kaiser *et al.*, 2014), with a parallel research emphasis on the formal contents of teacher education programs (e.g., Rowland & Ruthven, 2011; Depaepe, Verschaffel, & Kelchtermans, 2013; Tattó, 2013). A second school of thought, and the one that is developed in this writing, is that the most important competencies tend to be tacit, like skills involved in playing concert piano, learned but not necessarily available to consciousness.

## 1. Experts who can think like novices

This tacit, embodied dimension of teacher disciplinary knowledge has been the focus of my research and my teaching for the past 15 years. In this article, I describe strategies that have been co-developed with teachers to render elements of this knowledge more available in the moment of teaching —that is, in the phrasing introduced in the quote from Baumert *et al.*, above, to find ways to activate inert disciplinary knowledge.

Put in quite different terms, rather than attempting to specify and catalogue competencies to be mastered by individuals, I have focused on the development of structures and strategies to support collectives of pre-service and practicing teachers to deconstruct, interrogate, and elaborate their mathematics. These strategies are organized around the notion

that *the effective mathematics teacher is an expert who can think like a novice*. That is, teachers must be able to analyze and resynthesize their current mathematical understandings in ways that provide learners access to the metaphors, images, exemplars, and other elements that enable robust understandings. An assumption in this work, following research into the contrasts between experts' and novices' strategies of engagement (see Ericsson *et al.*, 2006), is that expert knowers tend to forget the difficulties they encountered in discerning relationships, construing principles, and other activities associated with developing robust understandings of concepts. They also tend to lose track of the «pieces» (*e.g.*, metaphors, exemplars) that they have integrated into their understandings, particularly as those understandings become more abstract and general. Such forgettings are vital to powerful conceptualizations and rapid applications. However, they can also be sources of frustration when attempting to teach others.

By way of illustrative examples, children are typically exposed to four distinct interpretations of number, about a dozen interpretations of odd/even, and more than two dozen interpretations of multiplication before leaving elementary school. When asked directly about their knowledge of these interpretations, teachers can typically summon only a few (Davis, 2011). Yet when observed in their classrooms, these same teachers can spontaneously draw on a broad spectrum of interpretations (Davis & Renert, 2014) —highlighting that even though they cannot explicitly identify key components of their understandings, they are able to enact them when

teaching. The issue here is that such enacted elements tend to be incidental and non-explicit. That is, teachers can invoke these diverse instantiations without being consciously aware of the dissonances that might be triggered for novices when, for example, multiplication is characterized as grid-making, scaling, hopping, and repeated addition —all in the same lesson.

The conception of teachers' knowledge at the heart of this work is profoundly influenced by Shulman's (1986) construct of *pedagogical content knowledge* (PCK), which he characterized as follows:

for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations —in a word, the ways of representing and formulating the subject that make it comprehensible to others. (p. 9)

For Shulman, PCK encompassed awareness of both established content and the processes by which content was established:

It requires understanding the structures of the subject matter. ... The structures of a subject include both the substantive and the syntactic structures. The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established. (p. 9)

When introduced, the construct of PCK sparked renewed interest in teachers' disciplinary knowledge within the English-speaking mathematics education research community, as it was quickly recognized as a way to explain the difficulty of finding correlations between teachers' formal background in mathematics and their students' performances on achievement tests. Notably, such thinking was already well developed in a significant body of research that was reported in several other languages, and which was perhaps most prominently associated with the notion of didactiques/didaktiks. Worthy of particular mention in this regard in Freudenthal's (1983) *Didactical Phenomenology of Mathematical Structures*, a lengthy and detailed exploration of didactical aspects of mathematical concepts that aligns well with Shulman's more general formulation of PCK. Freudenthal's didactical phenomenology extended beyond a study of mathematical structures, and examined mathematical «objects» as teachable and learnable forms. This didactical emphasis on mathematical knowledge entailed:

knowledge of mathematics, its applications, and its history. ... how mathematical ideas have come or could have come into being ... how didacticians judge that they can support the development of such ideas in the minds of learners ... understanding a bit about the actual processes of the constitution of mathematical structures and the attainment of mathematical concepts. (p. 29)

Paraphrasing Freudenthal: teachers' mathematics involves vastly more than con-

solidated knowledge of formal propositions. As one delves more deeply into the question teachers' mathematics, one finds ever more nuances, aspects, layers, and enactments.

To be explicit, and blending Shulman's and Freudenthal's conceptions of qualities of teachers' disciplinary knowledge, this work proceeds from a conviction that teachers' mathematics might be construed as *a way of being with mathematics knowledge that enables a teacher to structure learning situations, interpret student actions mindfully, and respond flexibly, in ways that enable learners to extend understandings and expand the range of their interpretive possibilities through access to powerful connections and appropriate practice*. That is, this work is concerned with affecting the «how» of classroom practice by addressing teachers' dispositions toward mathematics and mathematics learning. Collected together, these strategies that have been developed constitute «concept study,» a participatory methodology through which teachers interrogate and elaborate their mathematics (Davis & Renert, 2014).

Concept study is an evolving form. At present, it comprises a handful of intersecting strategies, each of which is subject to constant revision and elaboration. In fact, the aspects of concept study are not as much «strategies» as they are «emphases» —that is, they are sites to focus attentions as prospective and practicing teachers work together to identify key instantiations for concepts, interrogate the entailments of those instantiations, map out where those instantiations are introduced, integrate them into more coherent and

robust concepts, and use those emergent understandings to address persistent and vexing learning issues among students. By way of brief introductory example of its potential impact on teachers' knowledge, in a recent concept study of «zero» involving 11 secondary teachers, participants identified dozens of different representations of zero in grade-school mathematics curriculum, analyzed how and when those representations are introduced and used, and developed a consolidated «definition» of zero that they felt retained sufficient nuance to be pedagogically useful (see Davis & Renert, 2014, for a more complete account). One result in this collective effort was the realization that, within local programs of study, the concept of zero emerges through three major elaborations: *the counting zero*, *the measuring zero*, and *the systemic zero*. To truncate the group's extensive discussions, these zero-types were described as follows:

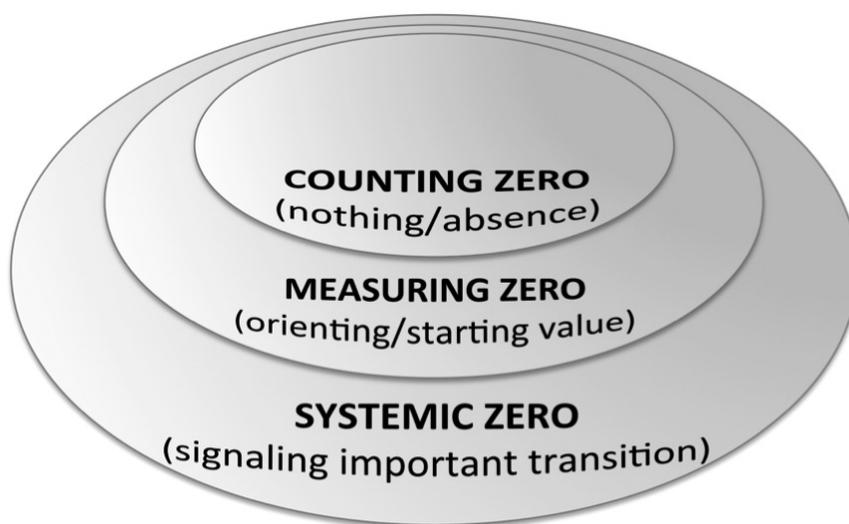
- counting zero –principal meanings of «nothing» or «absence,» prevalent in the early grades;

- measuring zero –principally serving as an orienting or starting value, associated with location– and movement-based interpretations of number, and prevalent in the middle grades; and

- systemic zero –arising in algebra and other high school applications, principally serving as an important transition in an object under scrutiny (e.g., the «zero of a function»).

Importantly, these meanings were not seen as distinct, but as emergent –that is, each transcending but including previous realizations. To highlight this insight, the group offered the nested image presented in Figure 1.

FIGURE 1: *Making sense of the concept of zero, as encountered in grade-school curriculum.*



Despite the fact that it took considerable time, conversation, and argument to arrive at this formulation, there was an interesting collective response when the nested image was finally drawn on the whiteboard. Voiced by one participant, and meeting with nods from around the room: «But ... we really already knew all that.»

This indication that they «already knew» was both surprising and expected. It was surprising because this consolidated formulation clearly had not been available to them at the beginning of the process. In fact, at the start, several participants argued that zero would not be a worthy topic for concept study since «its meaning is obvious.» Conversely, the claim that they «already knew» was to be expected because each teacher was able to draw on every aspect of this metarepresentation in her or his practice. Each arrived with a rich and integrated –but mainly tacit– knowledge. That is, each participant did indeed already know, but it was only through concept study that that expert knowing was reformatted in a way that made it more available to themselves, and thus for the learners/novices in their classrooms.

Over the past five years, concept study has moved beyond a research focus to become an integral element of teacher preparation at my home institution, which has afforded increased opportunity to observe closely how teachers might be supported in their efforts to reconfigure their expert knowledge in ways that render it more available when teaching. In the balance of this writing, I describe concept study and its impacts through a recent instance on the concept of *function*.

This experience involved 11 individuals who were advancing their studies of mathematics pedagogy.

## 2. Emphasis

### 2.1. Realizations

The term *realizations* is borrowed from Sfard (2008) and is used to refer to what might also be described as the «meanings», «interpretations», and/or «instantiations» of a concept. Briefly, as Sfard explains, the notion of realizations is used to collect all manner of associations that a learner might draw on and connect in efforts to make sense of a mathematical construct. More precisely, a realization of a signifier  $S$  refers to «a perceptually accessible object that may be operated upon in the attempt to produce or substantiate narratives about  $S$ » (p. 154). The distinction between a signifier and a realization is often blurred, as mathematical realizations can often be used as signifiers and realized further. Among many possible elements, realizations might draw on:

- formal definitions (*e.g.*, exponentiation is repeated multiplication)
- algorithms (*e.g.*, perform exponentiation by multiplying repeatedly)
- metaphors (*e.g.*, exponentiation as explosive growth)
- images (*e.g.*, exponentiation illustrated as branching of branches)
- applications (*e.g.*, exponentiation used to calculate interest)

— gestures (e.g., exponentiation gestured in an upward, parabolic sweep).

To be clear, the point of being attentive to a diversity of realizations is not to home in on any realization in particular; nor is it to identify them as right, wrong, adequate, or insufficient. It is that personal understanding of a mathematical concept is an emergent form, arising in the complex weaves of such experiential and conceptual elements. With regard to teachers' knowledge, realizations might be seen as the «objects» or «agents» of the complex system of teachers' knowledge of mathematics.

Notably, there is considerable research into the importance of realizations of grade-school mathematics concepts (e.g., Janvier, 1987, English, 1997; Goldin & Janvier, 1998; Lakoff & Núñez, 2000; Hiebert *et al.*, 2003). With this research backdrop, most teachers that I encounter are familiar with the suggestion that mathematical concepts have multiple interpretations, and my experience is that most teachers greet the task of identifying realizations as a sensible starting place in concept study.

The investigation of function thus began with the identification of key realizations of the concept across the curriculum. This was a brainstorming exercise. No attempt was made to rank or cluster realizations. Rather, the aim was to identify as many as possible. For the most part, participants drew from their own experience, but they also consulted programs of study and classroom resources. After roughly a half hour of engagement, they had generated the following list:

— curve that satisfies the vertical line test

— black box

— input/output process

— independent/dependent relationship

— 1-to-1 relationship/correspondence

— relationship between manipulated and responding variables

— equation with two or more variables

— indexed pattern

— mapping rule

— table of values

—  $f(x)$

Even though it had taken some time to generate this list, none of its entries came as a surprise to participants. Rather, additions tended to be met with such expressions as, «Oh yeah,» and «I'd forgotten about that.» That is, this exercise was truly one of recovery — of re-activating already-achieved insights. To put it in different terms, and to re-emphasize a vital quality of teachers' knowledge of realizations, the initial difficulty in identifying realizations was proven to be more about *not having immediate conscious access to what is known* than with *not knowing*.

The disorderly nature of this list is also instructive. While it was certainly the case that mentions of particular realizations sparked immediate mentions of others —suggesting close associations of those realizations in participants’ understandings— for the most part entries were added in a somewhat random manner.

It is precisely such randomness that prompted a different group of pre-service and practicing teachers some 15 years ago to develop a second emphasis, namely organizing such 1-dimensional lists into 2-dimensional landscapes.

## 2.2. Landscapes

There are dramatic differences of conceptual worth among realizations. Some can reach across most contexts in which a learner might encounter a concept —for example, a function is a «relationship between manipulated and responding variables.» Others are situation-specific or perhaps even learner-specific —such as a function is a «black box.» For us, this insight invited the question of how realizations relate to one another, which in turn compelled a landscapes-strategy to organize and contrast the entries on assembled lists of realizations.

The landscape emphasis was invented several years by a group of teachers who were frustrated with the incoherence of a raw list of interpretations for a concept (see Davis & Renert, 2014). In brief, virtually any distinction that has been used by educators might serve as a tool to create a landscape. The following are among

those that have proven particularly useful for teachers engaged in concept study: grade level, types of applications, discrete vs. continuous, conceptual vs. procedural, Bruner’s (1966) typology of enactive/iconic/symbolic representations, and Lakoff and Núñez’s (2000) four grounding metaphors of arithmetic (i.e., object collection, object construction, measuring stick, and motion along a path).

The second day of our concept study of function did not start out smoothly. Because the concept is covered explicitly only in high school, most of the primary- and middle-school teachers in the group were at first a little at sea at mentions of «black box,» «input/output,» «vertical line test,» and so on were offered by their high-school counterparts. Things became even more sluggish as the group moved to the landscape emphasis. In particular, given that there is no explicit mention of function prior to high school, the group struggled to find connections between early, middle, and senior grades. Nevertheless, they persevered —enabled in large part by the insistence of one lower-grades teacher that higher-grades counterparts explain in detail every one of their realizations so that connections to foci at the elementary level might be identified. It took most of the hour-long session, but eventually the group created a landscape that revealed a flow across three major topics of study: pattern à equation à function. The group elected to illustrate the insight as a nested, emergent flow across the grades (see FIGURE 2) that also highlighted the movement from discrete applications to continuous.

FIGURE 2: A landscape of how the concept of «function» unfolds.

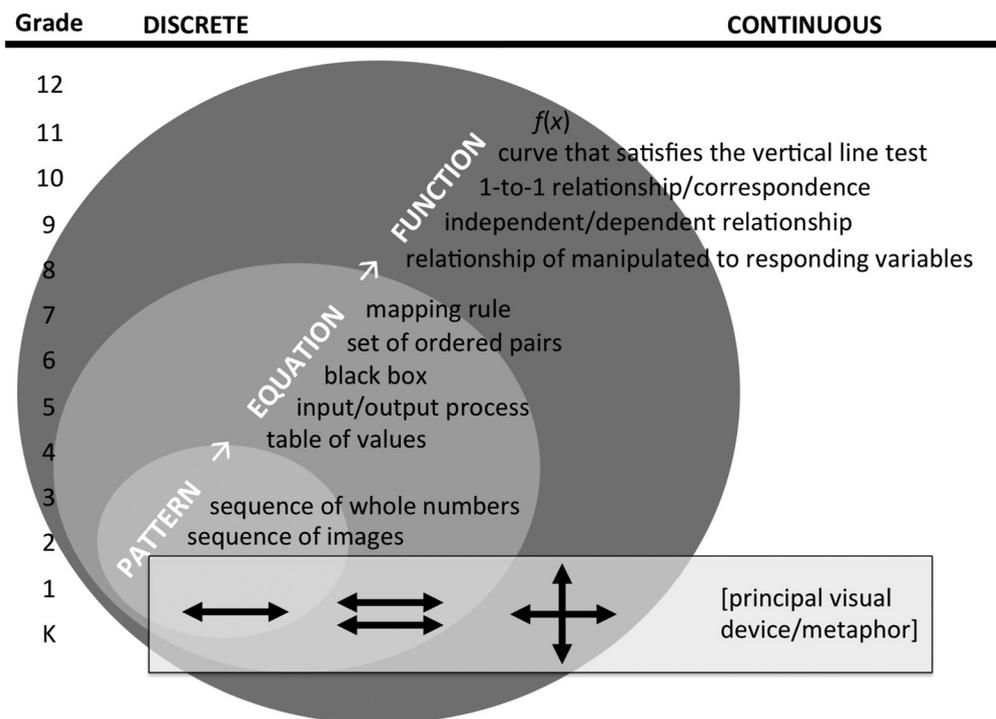


Figure 2 illustrates that the landscape emphasis is not only as descriptive but also as creative. For us it was a clear instance of the emergence of novel mathematical knowledge. The collective generated insights that were not *explicitly* available to any its members prior to the engagement, but the eventual product depended entirely on the combined knowledge of all participants. That is, the knowledge *was there*, but it was inaccessible in its entirety to any individual, partly because of its largely tacit nature and partly because aspects of it were held by different members of the group. These partly-tacit-partly-distributed insights were given voice and collected in the discussion that followed. They included the following:

- Rather than being introduced in the high-school years, the concept of function is actually distributed across all the grades, with significant groundwork in the very early years.

- While the unfolding of the concept across the grades was now clear, it was evident that it was not presented in a coherent manner. Elementary level teachers had been teaching «patterns» as an isolated, disconnected topic. (In fact, one confessed to ignoring the topic, since she could not see its relevance.) Conversely, secondary teachers had been addressing «function» as a new construct and none attempted to link the topic to student learning prior to 8th-grade algebra.

– Each of the three core topics –i.e., patterns, equations, and functions– is associated with a distinct image, and there is a clear developmental trajectory across these images. As denoted in the box in the lower part of FIGURE 2, «patterns» are associated with *single, directional lines*, and are typically introduced as invitations to find the rule that governs rows of shapes or numerals. «Equations» are most associated with *parallel, directed lines*, as typically introduced with T-tables, ordered pairs, and similar forms that couple an index to a generated value. «Functions» are most often associated with a *pair of perpendicular lines* (i.e., a coordinate system), leading to two-dimensional representations.

The last of these points invited considerable discussion, as it was a new idea to every participant in the study. In fact, one question that arose immediately was whether the sequence of «single line à parallel lines à perpendicular lines» was implicit across other elements of the curriculum –and the answer turned out to be «yes.» Multiplication, for instance, is typically introduced through linear interpretations (e.g., repeated addition, hopping along a number line), then in terms of parallel lines (e.g., stretching/compressing number lines), and then in two dimensions (e.g., area making, linear function).

Such discussions served to frame the next day's emphasis, in which each realization was analyzed for its strengths and weaknesses –as both a tool of thinking and a device to enable learning.

### 2.3. Entailments

Each realization of a concept carries a set of logical implications and entailments. The intention of this third emphasis is to study how different realizations shape the understanding of related mathematical concepts (e.g., how the «vertical line test» might enable or constrain understandings of function). In the process of exploring different entailments, participants are compelled to consider mathematical concepts afresh and not only in well-rehearsed ways. Some surprises emerge.

In the context of concept studies, we refer to the type of thinking within this emphasis as «substructuring.» The word *substructuring* was first used in the context of a concept study by the a teacher who was unhappy with my use of the word *unpacking*, which is one of the more popular notions to have arisen in recent discussions of teachers' disciplinary knowledge of mathematics (e.g., Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008; Ma, 1999). It is used to describe an aspect of mathematics knowing that is unique to teachers. Whereas an important component of research mathematicians' work is to collect their thinking into compact formulations –that is, packing– it is teachers' task to perform the reverse operation. Teachers must be able to take apart formulae, operations, and mathematical terms, so that students can gain access to the thought processes and ideas that they represent.

Concept studies inevitably include some unpacking activities. For example, as illustrated by the «Realizations» emphasis, when seeking to make sense of a complex mathematical idea, one useful starting

point is to generate lists of metaphors, analogies, and images that might be associated with that idea. The process of generating such lists both renders explicit the primarily analogical substrate and the primarily tacit nature of human knowing. The principal aim of unpacking activities within concept studies is to recall the figurative aspects of understanding, which expert knowers might have forgotten they know.

However, subsequent emphases in concept studies cannot be aptly construed in terms of unpacking. Deep understanding of a concept requires more than pulling apart its constituent parts; it also entails examinations of how these parts complement and contradict one another in different contexts and circumstances. Teachers' engagements in such examinations often lead to the generation of novel insights that transcend pre-established and recovered packed insights. This is precisely the sense intended by the teacher who offered the term: the word *substructing* suggested a sense of dismantling and rebuilding. For him (and, very quickly, for the group), it highlighted the creative dimensions that inhere in the reworking process and distinguish them from the merely descriptive/interpretive emphases of unpacking.

It turned out that his meaning of dismantling and rebuilding is very close to the dictionary definition of the word. *Substructing* is derived from the Latin *sub-*, «under, from below» and *struere*, «pile, assemble» (and the root of *strew* and *construe*, in addition to *structure* and *construct*). To substruct is to build beneath something. In industry, *substruct* refers to reconstructing a building without demolishing it —

and, ideally, without interrupting its use. Likewise, in our concept studies, teachers rework mathematical concepts, sometimes radically, while using them almost without interruption in their teaching.

Most often within concept studies, the work of substructing the entailments of different realizations is experienced as tedious and frustrating — which is to be expected. Humans are mainly associative thinkers, and for the most part this associational work occurs beneath the level of explicit awareness (Lakoff & Johnson, 1999). Such difficulties were certainly encountered by teachers in their efforts to substruct the entailments of different realizations of function, the results of which are presented in Figure 3.

There is a great deal of information presented in Figure 3. I have included it in its fullness to illustrate and underscore the extent of tacit knowledge that secondary teachers must have in order to draw on this diversity of instantiations in their teaching. Indeed, much more could be said, but I will limit myself to remarking specifically only on the right-most column. That column comprises participants' insights into the conceptual limitations of varied realizations. Pedagogically speaking, in the context of our meeting, it was easily the most important and generative portion of the entailments chart — mainly because it afforded the teachers opportunity to think through how devices that are intended as learning aids might actually invoke some unintended and misleading associations, but partly because it compelled each participant to rethink at least some aspects of their knowledge of functions.

On the latter point, as it turned out, this column afforded participants opportunities to make sense of what a function is by making it clear what a function is not.

FIGURE 3: An entailments chart for the concept of function.

If a function is defined in terms of...	...then the type of data is...	... then it is represented as...	... then in elementary school it looks like...	... then in middle school it looks like...
<b>Indexed Pattern</b>	Discrete	Sequence	Growing patterns	Sequences
<b>Creating a Table of Values</b>	Discrete	Horizontal or Vertical T-Chart	T-Chart	T-Chart
<b>Input/output</b>	Discrete	Ordered Pairs	Sequenced practice - counting	Using linear equations to find missing values; Ordered pairs
<b>Black Box</b>	Discrete	Two sets of Data	Missing value in an equation	Missing function problems
<b>Mapping</b>	Discrete	Arrow Diagram	Classification	Transformational arrows
<b>Equation with 2 or more variables</b>	Discrete/ <b>Continuous</b>	Equation (implicit; $y=...$ )	Multiple-representation problems —how many perimeters can you show of 24	Equations (implicit; $y=$ )
<b>Manipulating/Responding Variables</b>	Discrete/ <b>Continuous</b>	Physical Event à collect data à graph à trend seeking à interpolate/ extrapolate	Experimental data —growth chart	Experimental data graphing; interpolate/extrapolate
<b>Independent/Dependent Relationship</b>	Discrete/ <b>Continuous</b>	Rule	Pattern rules	Linear equations; graphing
<b><math>f(x)</math></b>	Discrete/ <b>Continuous</b>	Equation (explicit; $f(x)=...$ )	N/A	Equation (implicit $y=$ )
<b>1:1 Relationship</b>	<b>Discrete/</b> Continuous	Anything in this column	Counting based 1 to 1 correspondence	Ordered pairs; T-chart;
<b>Vertical Line Test</b>	Discrete/ <b>Continuous</b>	Graph	N/A	N/A

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If a function is defined in terms of...	... then in high school it looks like...	...then the realization is...	Conceptual Limitation(s)
<b>Indexed Pattern</b>	Sequence	enactive, iconic and symbolic	Implication of Uniqueness; for every $i$ , there is a unique $n$ ; dependence on initial position, or starting point; requires a starting point; domain is always natural numbers
<b>Creating a Table of Values</b>	T-Chart	iconic and symbolic	Orientation of Data; Idea of discreteness; Possible for left column to be random; Implication of Uniqueness (all of the limitations?)
<b>Input/output</b>	Ordered Pairs	iconic and symbolic	Piecemeal Production; missing the whole
<b>Black Box</b>	Two sets of Data	iconic	Magical Mystery
<b>Mapping</b>	Arrow Diagram	iconic and symbolic	Implied container schema
<b>Equation with 2 or more variables</b>	Equation (implicit; $y=...$ )	symbolic	Some equations with 2 or more variables may not be functions eg. $y=\pm(\sqrt{x})$
<b>Manipulating/Responding Variables</b>	Physical Event à collect data à graph à trend seeking à interpolate/ extrapolate	enactive à iconic à symbolic	Contrived and controlled and contextual
<b>Independent/Dependent Relationship</b>	Rule	symbolic	Suggests they are not reversible
$f(x)$	Equation (explicit; $f(x)=...$ )	symbolic	Notation implies multiplication; coordinates are $(x, f(x))$ they don't get that $f(x)$ is $y$
<b>1:1 Relationship</b>	Anything in this column	enactive, iconic and symbolic	Implication of uniqueness; for every $x$ , there is a unique $y$
<b>Vertical Line Test</b>	Graph	enactive and iconic	Rejects some functions when $x=f(y)$ ; includes others that are not function

The chart also triggered a discussion of the criteria teachers should use to se-

lect realizations. Starting with an agreement that such criteria of typically impli-

cit and uninterrogated, group members soon assembled a list of considerations when making these decisions:

- mathematical appropriateness
- appropriateness for the situation at hand
- teacher familiarity with the realization
- student familiarity with the realization
- connection to previous realizations
- conceptual reach/mathematical power
- potential for elaboration
- danger of unintended and/or misleading associations.

The session closed with the reflection that teachers' capacity to move fluidly among realizations was not so much a matter of «having a firm handle» on those realizations, but rather a consequence of losing the ability to differentiate among them.

#### 2.4. Blends

The three emphases described so far—realizations, landscapes, and entailments—are focused mainly on making fine-grained distinctions among interpretations. Not surprisingly, while the participating teachers showed strong interest in these emphases, they also voiced some

frustrations as the shared work unfolded. «Function» is, after all, a mathematically coherent concept, not an assemblage of images and implications that can be laid out in discrete pieces. The blending emphasis is intended to address this concern. It is about seeking out meta-level coherences by exploring the deep connections among realizations and by assembling those realizations into grander, more encompassing interpretations that yield further emergent interpretive possibilities.

For this emphasis, we drew principally on cognitive science research into conceptual blends (Fauconnier & Turner, 2002), which examined the emergence of new and more powerful discursive objects through combinations and mash-ups of existing ones. In particular, following diSessa (2004), we introduced conceptual blends to the cohort in terms of *metarepresentations*. As diSessa described, metarepresentational skills comprise «modifying and combining representations, and ... selecting appropriate representations» (p. 296), subcomponents of which include inventing and designing new representations, comparing and critiquing them, applying and explaining them, and learning new representations quickly.

Elsewhere (Davis & Renert, 2014), several methods for supporting teachers in their efforts to generate useful and mathematically sound blends have been described. Within the concept study of function described in this writing, the strategy used was to move back and forth between individual consolidation and group discussion. Specifically, after the entailments chart (Figure 3) had been assembled, the

suggestion was made that each person should write out their own answer to the question, «What is a function?» on a sheet of paper and then post their responses on the wall. When that was done, the group proceeded to cluster responses according to core theme, and three emerged: function as relationship, path, and value-assigner.

The participants then assigned themselves to one of three subgroups that formed around these themes, and each subgroup took on the task of pulling together a cluster of thematically similar interpretations into a consolidated description of function. The three consolidated descriptions were then contrasted, and the group undertook another consolidation effort. After a few such iterations, the collective response to the question, «What is a function?» was:

**Function** —a relationship (articulated as a mapping rule, ordered pairings, axes-associating curve, transformation process, etc.) between two sets of values, in which each value in the source set («the range») corresponds to at most one value in the target set («the domain»).

I intervened at that point to request a more succinct version, and the following was generated after one more round of discussion:

**Function** —a relationship in which each specified domain value has at most one range value.

It is important to note that this emergent definition was not assembled for dissemination. It was a product of the group, intended

for the group. And meaningful to the group. As one lower-grades teacher commented, «I didn't give a rip about functions when we started. Now not only do I know what they are, I actually care about them.»

Appreciating these details is critical to understanding the potential contributions of concept-study-like activities with teachers. Arguably there is nothing remarkable in their final formulation, above. It very much resembles standard definitions of function in classroom resources and curriculum guides —and, indeed, in a blind review of a related report on this incident, one commentator criticized, «It would appear that several sessions were wasted, given that the final result could have been generated at the start by looking up 'function' in a mathematics dictionary.»

Unfortunately, that commentator missed the detail that such formal definitions are not always appropriately meaningful to teachers —including teachers who actually recite those definitions in class. The point of the work was not to generate this summary definition; it was to appreciate the breadth of experience and interpretation that it conceals, thereby opening up pedagogical possibilities.

Such pedagogical possibilities are precisely the focus of the fifth emphasis of concept study.

### 2.5. *Pedagogical Problem Solving*

«Pedagogical problem solving» represents a move into some of the more complex processes entailed in teaching mathematics, one that goes beyond the study of

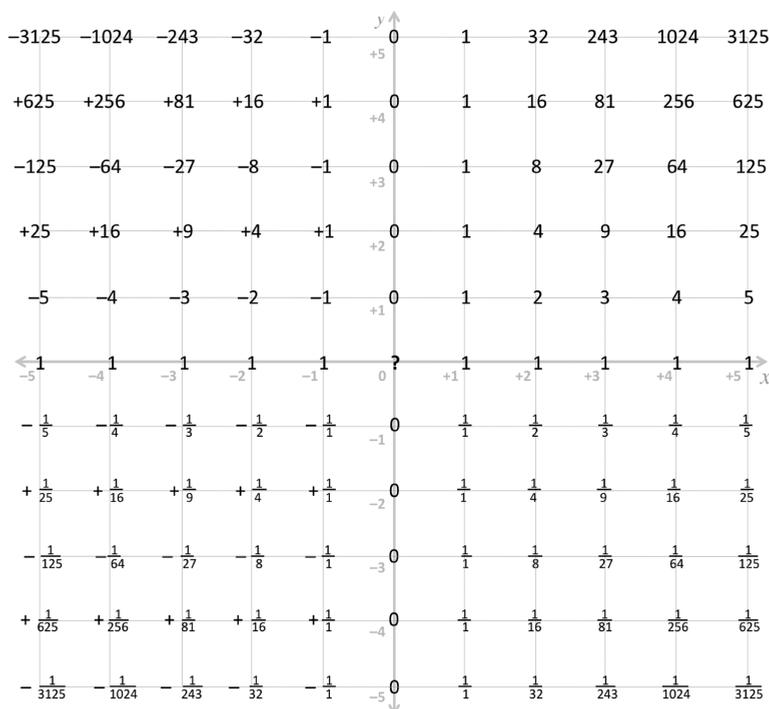
discrete concepts. This emphasis is, in a very deep sense, a site of the *real mathematical work of teachers*, as it focuses on the mathematical problems that they encounter daily and that are specific to their profession. Unlike earlier emphases, which tended to artificially circumscribe mathematical concepts and meanings in ways that are not likely to be implemented in mathematics classrooms, this new emphasis is developed around the actual questions that meaning-seeking learners ask.

Pedagogical problem solving capitalizes on the interpretive potentials that arise collectively when teachers draw on various instances of individual expertise in order to broach perplexing problems of shared interest. It situates the enterprise

of concept study in the everyday complexities of mathematics teaching and problem solving, where multiple concepts are at play. It is in such situations that the immensity and emergent possibilities of mathematics for teaching can come into focus.

Very often, the questions addressed within this emphasis diverge somewhat from the concept that served as the focus of the previous emphases. Such proved to be the case in this instance, when the problem that dominated the session was how a 9th-grade teacher might help students better understand exponentiation. Several pieces of advice arose, two of which are highlighted here. They were selected because they reflect key insights developed during the concept study.

FIGURE 4: A piece of an exponentiation grid,  $x^y$ .



The first was the suggestion to build an «exponentiation grid» (see Figure 4) that was a blend of a multiplication table and a Cartesian grid. The grid format permitted values to be entered for positive and negative bases and exponents –which in turn, participants reasoned, should support efforts to discern patterns and illustrate relationships. The trigger for this suggestion was the realization that experiences that contribute to the concept of function unfold in a «single line à parallel lines à perpendicular lines» sequence. As the group’s reasoning went, the two-dimensional might be deliberately used as a tool of analysis.

As reported elsewhere (Davis, 2014), the grid did indeed have precisely that effect when used in a classroom –and for reasons that are instructive. There is a vital difference between concepts studied at elementary and secondary levels. Whereas almost all the concepts encountered at the elementary level can be interpreted in terms of (i.e., are analogical to) objects

and actions in the physical world, the analogies for concepts at the secondary level are mostly mathematical objects (see Hofstadter & Sander, 2013). Making analogies, then, is both a mechanism for extending mathematical insight and a window into the structure of mathematics knowledge.

Aware of this detail, the teachers in the concept study group also suggested a second activity for the classroom –namely an entailments– like chart (triggered by their own chart) that might be used as a tool to invite students into examining the relationships among addition, multiplication, and exponentiation. The intention here, that is, was to deliberately invoke learners’ irrepressible tendency to seek out and/or invent associations among experiences. Figure 5 shows the results generated by one class in this regard. (Note that participants adopted a vertical arrow notation for powers, rather than the standard superscript notation, in order to highlight analogies to prior operations.)

FIGURE 5: *Conjectures for exponentiation based on analogies to addition and multiplication.*

Topic / Property	How it looks for addition (+)	How it looks for multiplication (x)	Speculation for exponentiation ( )	T/F
<b>Commutative Property</b>	$a + b = b + a$	$a \times b = b \times a$	$a b = b a$	FALSE 23 ≠ 32
<b>Reverse operation</b>	Subtraction (-)	Division (÷)	De-exponentiation ( )	
<b>Identity element</b>	0 ... as in $a + 0 = 0 + a = a$	1 ... as in $a \times 1 = 1 \times a = a$	1? ... since $a 1 = a$ ... although $1 a = 1$	



Topic / Property	How it looks for addition (+)	How it looks for multiplication (×)	Speculation for exponentiation ( )	T/F
<b>Inverse values</b>	Additive inverse of $a$ is $0 - a$ , or $-a$ ; $a + (-a) = 0$	Multiplicative inverse of $a$ is $1 \div a$ , or $;$ $a \times = 1$	Exponentative inverse of $a$ is $1 a$ , or $a$ ; $a (a) = 1$	
<b>Operating on the opposite</b>	Subtraction can be done by adding the [additive] inverse: $a - b = a + (-b)$	Division can be done by multiplying the [multiplicative] inverse: $a \div b = a \times$	De-exponentiation must be doable by exponentiating the [exponentative] inverse: $a b = a (b)$	
<b>«Next» operation</b>	A repeated addition is a multiplication.	A repeated multiplication is an exponentiation.	A repeated exponentiation must be a ... something.	
<b>«Next» set of numbers</b>	When you allow subtraction, you need signed numbers.	When you allow division, you need rational numbers.	When you allow de-exponentiation, you need another set of numbers.	

Once again, much could be said here. Of perhaps greatest significance is the obvious and profound impact of teachers' collective concept study on one participant's individual practice. For him, the experience of engaging in a concept study of function transformed how he framed other topics —as moments of inquiry that invited pattern noticing, speculation, and justification.

### 3. Discussion and conclusions

As noted earlier, there is no consensus on the nature of teachers' disciplinary knowledge of mathematics, let alone how that knowledge might best be developed and exercised. The intention of this article is thus not to argue that much thinking has been misguided and much research has been misdirected; it is, rather, to prompt attentions toward aspects of expert knowing that cannot be readily taught and examined.

To that end, the point of developing this report around a concept study of «function» was not to identify facts that teachers must know. Indeed, it would be unsettling if specific details identified by participants in the concept study reported here were to find their way into textbooks and examinations for teachers. Rather, the point of presenting those details was to illustrate the power of concept study for simultaneously revealing the complexity of mathematical ideas and developing knowledge of mathematics for teaching.

As the results of the emphasis of pedagogical problem solving (i.e., the preceding section) highlight, discussions typically spill past the explicit topic of the concept study. And, as those results also foreground, participants are typically compelled to grapple with topics that go well beyond facts and procedures. For example, it turns out that many of the

extensions and speculations about exponentiation developed by students (i.e., in the «exponentiation» column of Figure 5) are actually problematical —and the realization of that detail presses teachers to engage with such topics as mathematical justification, the powers and the perils of analogy, the associative (vs. logical) nature of human thought, and so on.

Some might argue that the incorrectness of some of the mathematical speculations presented in Figure 5 would be a compelling reason to avoid this sort of activity. For the teacher in this episode, however, it proved an opportunity to engage with such fundamental issues as «how mathematics knowledge is generated,» «the nature and role of proof,» and «the nature of mathematics research.» It also presented an occasion for serious examination of how people learn —and, in particular, how the habits of making associations and generalizing from one situation to another can lead to major problems in understanding.

In other words, such emphases in teacher education can present opportunities to integrate a range of competencies that have recently been identified as distinct elements of teachers' disciplinary knowledge (see, e.g., Ball, Thames, & Phelps, 2008). The suggestion here would be that, while there is clear value in identifying and studying sub-elements of teachers' mathematics, these elements must be recognized to arise spontaneously and simultaneously in teaching. Arguably, then, experiences during teacher preparation should present at least some opportunity for similarly integrated encounters.

More important, perhaps, is the need to affect prospective teachers' dispositions toward attending to their students' sense-making —that is, to the ways that learners might be knitting diverse realizations together with idiosyncratic experiences. As reported elsewhere (Davis & Renert, 2014), there is abundant evidence that concept study not only supports but compels teachers to attend to their students in ways that are focused on the realizations and blends that are emerging. At the same time, concept study affords strategies to bring to bear to distinguish between powerful realizations and less useful ideas —in the process, enabling teachers to make strong decisions about what to underscore and what to ignore as new ideas are presented in the classroom.

Finally, with specific regard to the preparation of secondary mathematics teachers, a consistent «outcome» of concept studies is that engaging elementary and secondary teachers together in critical examinations of the subject matter can be highly productive. With the much broader range of realizations that are likely to be implicit in secondary students' understandings, high-school-level concepts can be very difficult to substruct. However, that task can be greatly eased by considering high-school concepts as extensions of learning that begins in the early years.

Such trans-level appreciation of mathematics learning appears to be a vital aspect in preparing teachers to be experts who are able to think like novices. Conceptual fluency in *using* mathematics is rooted in forgetting-through-blending of the many realizations of a concept one has

encountered. Arguably, conceptual fluency in *teaching* mathematics is tethered to a remembering-through-substructuring those realizations.

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### Summary: The mathematics that secondary teachers (need to) know

Through an extended example, I explore the relevance for secondary mathematics teachers of «concept study» —a mode of working with pre-service and

practicing educators designed to support the development their understandings of mathematics in ways that help to activate their formal disciplinary knowledge when teaching.

**Key Words:** Mathematics education, teachers training, secondary school teachers, concept study.

### Resumen:

#### Las matemáticas que los profesores de educación secundaria conocen (o necesitarían conocer)

A pesar de la gran cantidad de investigación que se ha dedicado al tema de los conocimientos disciplinares de los profesores de matemáticas, el mismo está aún lejos de poder considerarse un constructo bien formulado. Este déficit plantea problemas, habida cuenta de los efectos que las suposiciones acríicas y las prácticas arraigadas sobre el tipo de matemáticas que los profesores deberían

saber, en relación con los programas de formación docente, especialmente en educación secundaria. A través de un ejemplo amplio, este artículo explora la relevancia, para los profesores de matemáticas de educación secundaria, del enfoque del «estudio de concepto» (*concept study*) una forma de trabajar en la formación inicial y permanente de los profesores que se centra en desarrollar su comprensión de las matemáticas, como modo de activar su conocimiento formal de la disciplina. Los resultados del estudio apuntan, entre otras conclusiones, a los beneficios de una mayor articulación entre la formación de los profesores de educación primaria y de educación secundaria, pues el tratamiento de conceptos matemáticos que es complicado abordar en educación secundaria, se facilita cuando son considerados extensiones de aprendizajes realizados durante los primeros años.

**Descriptor:** Formación matemática, formación docente, profesores de educación secundaria, estudio de concepto.