revista de EDUCACIÓN
Nº 370 OCTUBRE-DICIEMBRE 2015

Usos del error en la enseñanza de las matemáticas

Error uses in teaching mathematics

María José González López
Pedro Gómez
Ángela María Restrepo

GOBIERNO DE ESPAÑA
MINISTERIO DE EDUCACIÓN, CULTURA Y DEPORTE

Calidad de Revistas Científicas Españolas
FECYT | 2013

Error uses in teaching mathematics

Usos del error en la enseñanza de las matemáticas

DOI: 10.4438/1988-592X-RE-2015-370-297

María José González López
Universidad de Cantabria, Dpto. de Matematicas, Estadística y Computación
Pedro Gómez
Ángela María Restrepo
Centro de Investigación y Formación en Educación del la Universidad de Los Andes

Abstract

Learning errors are involved in many decisions adopted by mathematics teachers. Although some studies show that alerting teacher trainees to the use of error in teaching makes their students more aware of their learning and improves their performance, little research has been conducted on such use. This article identifies and describes the uses of error reported by 26 mathematics teachers (working in groups) participating in a two-year training programme for practising mathematics teachers. The groups recorded and justified the teaching decisions adopted when planning their approach to mathematical topics. A conceptual framework was developed in the context of this study to analyse teachers' decision-making in the context of such planning . Based on the notions of purpose, action and result, the conceptual framework was used in the construction of an initial set of categories in keeping with which the groups' final reports were coded. The categories were refined via a cyclical process in which the raw data were re-coded and the categories revised. The aforementioned purposes, actions and results were subsequently ranked in light of the outcome of that process. The uses of error were found to relate to three general purposes: overcoming error, evaluating students' cognitive knowledge and generating information useful for other lesson planning-related questions. Each of these uses was characterised in terms of the specific purposes associated with each general purpose, the respective actions and the results deriving from each action. An

Received: 24-04-2014 Accepted: 22-05-2015

understanding of these uses is deemed to be useful for the design and development of teacher training programmes.

Keywords: Decision-making, learning errors, mathematics, lesson planning, secondary education, teacher education.

Resumen

Los errores de aprendizaje están presentes en muchas de las decisiones que el docente toma en relación con la enseñanza de las matemáticas. Pero, las investigaciones sobre el uso del error en la formación de profesorado son escasas, aunque algunas de ellas muestran que la formación en este ámbito hace a los estudiantes más conscientes de su aprendizaje y mejora su rendimiento. En este artículo, identificamos y describimos los usos del error que 26 docentes de matemáticas de secundaria (organizados en grupos) manifestaron en el contexto de un programa de formación de profesorado de matemáticas en ejercicio de dos años de duración. Para ello, desarrollamos un marco conceptual sobre procesos de toma de decisiones del profesorado cuando planifica la enseñanza de temas de matemáticas. Este marco conceptual está basado en las nociones de propósito, acción y resultado. Con base en este marco conceptual, establecimos un conjunto inicial de categorías que nos permitió codificar los informes finales de los grupos de docentes. En estos informes, ellos registraron las decisiones que tomaron al elaborar sus unidades didácticas de matemáticas y las justificaciones que les llevaron a tomarlas. Realizamos un proceso cíclico de refinado de las categorías y de nueva codificación de la evidencia con las nuevas categorías para verificar su validez. Este proceso nos permitió establecer una jerarquía entre los propósitos, las acciones y los resultados. Encontramos que los usos del error se organizan según tres propósitos generales: superar el error, evaluar el estado cognitivo de los estudiantes y producir información útil para otros aspectos de la planificación. Caracterizamos cada uno de estos usos mediante los propósitos concretos asociados a los propósitos generales, las acciones asociadas a cada propósito y los resultados vinculados a cada acción. Consideramos que conocer estos usos resulta útil para el diseño y desarrollo de programas de formación de profesores.

Palabras clave: Errores de aprendizaje, matemáticas, toma de decisiones, planificación curricular, educación secundaria, formación del profesorado.

Introduction

Errors are inherent in any learning process (Borasi, 1994; Lannin, Barker y Townsend, 2007; NCTM, 2000; Rico, 1997). The literature is rife with research reports that identify frequent student errors in mathematical topics, explore the source of such errors and put forward proposals geared to helping pupils overcome them (Santagata, 2005; Son, 2013; Son and Sinclair, 2010). Helping students to surmount common errors is, then, one of a teacher's primary goals in planning lessons on mathematical topics. Error may be present in other mathematics teachers' decisions, however. This article contains a detailed description of the processes in which teachers use error in mathematical topic lesson planning and delivery. More specifically, it is based on a conceptual framework for decision-making that identifies the wide variety of uses to which error was put by groups of participants in a training programme for practising mathematics teachers.

The article begins with a discussion of the relevance of research on error in mathematics education and a justification of this study as a contribution to the present state of the art on the subject, followed by a description of decision-making viewed as a conceptual framework for characterising the uses of error in mathematics lesson planning. On that basis, a definition of use of error is put forward and the focus of the study is established. The context of the empirical study is then described, along with the sources of information used and the tools and procedures deployed to collect and analyse the information. The uses of error identified in the empirical study are subsequently introduced. Lastly, a series of considerations on the uses of error in mathematics teacher training programmes are addressed.

Justification and relevance

Learning theories envision error in different ways (Santagata, 2005). In behaviourist theory, for instance, error is regarded as a knowledge deficiency to be remedied by the teacher. Constructivism regards error as the result of applying knowledge to the wrong context. This vision of error has implications for teaching. The aim is for new knowledge to arise from situations in which subjects' knowledge is thrown off balance

Received: 24-04-2014 Accepted: 22-05-2015

because their cognitive structures do not allow them to broach the situation appropriately (Simon y Schifter, 1991, p. 310). Consequently, in education based on the constructivist vision of learning, the inference is that teachers may induce their pupils to err, help them see their error and generate a cognitive conflict that ultimately modifies their knowledge (An and Wu, 2012; Borasi, 1996; Brousseau, 2001; NCTM, 2000).

On the grounds of that constructivist approach to learning, several researchers have proposed error-based teaching strategies. Borasi (1994), for instance, suggested that errors can be used as springboards for inquiry and showed this strategy to have beneficial effects on students' learning. Schoenfeld (2011) proposed teaching by diagnosis in which teachers anticipate students' actions (including errors) and build their teaching on such predictions. Garuti, Boero and Chiappini (1999) devised a "voices and echoes" game to broach conceptual errors. Lannin et al. (2007) and Prediger (2010) proposed using errors as learning catalysts. These and other researchers have put forward a number of ways in which teachers could use error in education.

What does research have to say about teachers' use of error in everyday practice? While a few scholars have explored such use of error, the literature on this subject is patchy (Heinze and Reiss, 2007; Santagata, 2005). In her comparative study of U.S. and Italian teachers, Santagata established the following categories to characterise reactions to error. Teachers (a) furnished the right answer; (b) repeated the question to the student who erred; (c) re-stated the question, providing some helpful hints; (d) asked the student to explain how she deduced her answer; (e) used the above strategies with a different student; (f) asked the class to identify the error and suggest alternative answers; (g) chose the right answer from among the students' replies. She found that in Italian classrooms error was treated publicly, whereas in the U.S. teachers confined their explanation to the student involved. In both countries, the most frequent reaction was for teachers to correct the error themselves. The assistance furnished consisted primarily in simplifying the problem that induced the error. Italian teachers tended to help the student who erred, while U.S. teachers turned to another student.

Other studies have yielded similar results. Some showed, for instance, that teachers in the U.K. tended to protect students' self-esteem, teachers in France reacted directly to students' errors, and teachers in Japan addressed errors positively, turning them into a source of class discussion.

Schleppenbach, Flevares, Sims and Perry (2007) found that while U.S. teachers tended to avoid or conceal errors, Chinese teachers encouraged students to reflect on why they erred. Son and Crespo (2009) showed that future mathematics teachers tended to repeat procedures until students could spot the error, ignoring possible underlying conceptual considerations. In a more recent study, Son (2013) confirmed that finding, whose explanation, as Santagata (2005) suggests, would appear to lie in history- and culture-related differences (p. 493).

Very few studies have been published on training teachers to use error (Heinze and Reiss, 2007, p. 3-10). In their meta-analysis of the effects of teacher training on handling error, Keith and Frese (2008) found the effect to be significant and beneficial in the 24 studies they reviewed. In the area of mathematics education, Rach, Ufer and Heinze (2013) observed that training teachers to be error-tolerant had a beneficial effect on students' attitudes, although no significant effects on their cognition were found. The Heinze and Reiss (2007) study showed that training teachers to use error in the classroom made students aware of how to handle error and improved their performance. Brodie (2014), in turn, reported that working in learning communities furthered teacher trainees' ability to identify, interpret and handle errors (as suggested by Prediger, 2010) and that such work induced them to reflect on their own knowledge.

The present study, which lies within the general scope of the research described, characterises the use of error by mathematics teachers participating in a training programme.

Decision-making

Different approaches have been adopted in the extensive literature on decision-making, depending on the context of the problems involved (Hansson, 2005). While many such studies draw from economics and are based on statistical models (Savage, 1951), in disciplines such as medicine, the environment, politics or education, qualitative analyses prevail (Kahneman, Slovic and Tversky, 1982). Some of the most recent studies are geared to determining whether decisions are pre-established or solutions are tailored to situations as they arise. In environments with many constraints, decision-makers have been observed to tailor solutions to each specific situation (Stefaniak and Tracey, 2014).

Received: 24-04-2014 Accepted: 22-05-2015

Bishop (1976) defined teacher decision-making to be pivotal to education. Decision-making has also been analysed from different perspectives (Borko, Roberts and Shavelson, 2008). Any number of papers have analysed the factors involved in teachers' classroom decisions in highly interactive situations that call for an immediate and spontaneous reaction to the incidents arising in practice (see Schoenfeld, 2010, for instance). Others, however, have studied teachers' decisions from the perspective of their analysis of data on students' progress (Schifter, Natarajan, Ketelhut and Kirchgessner, 2014). These studies have frequently attempted to deduce implications for teacher training, particularly initial training (Call, 2012; Rich and Hannafin, 2008).

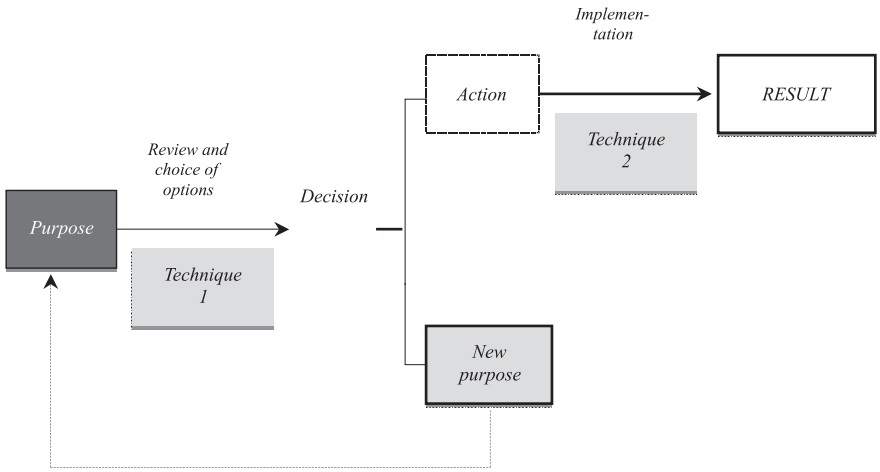
The present paper addresses teacher trainee decision-making when planning and reflecting, outside the classroom, on how to teach a given lesson. In that context, teachers engage in quiet, prolonged reflection. The structure of that process is described below.

Decision-making structure

A decision is the result of a cognitive process conducted by a person or group choosing one from a series of options to fulfil a given purpose. The flowchart in Figure I was inspired by the ideas of Bishop (1976), Shavelson, Webb and Burstein (1986) and Evans, Over and Handley (2003). It depicts the basic structure of teachers' decision-making during pre-classroom planning of how to teach a given lesson. It shows the key notions involved in the process: purpose, technique, decision, action and result, described below and illustrated with examples of the use of error. The process begins with teachers' purpose. To fulfil it, they have a series of options from which to choose. They make a decision when, deploying certain techniques, they adopt one of those options. Decisions may lead either to: (a) an action, in which case the teacher uses a technique to implement the action and obtain a result; or (b) a new purpose, more specific than the first, in which case the process begins anew.

Received: 24-04-2014 Accepted: 22-05-2015

FIGURE I. Basic decision-making structure and key notions



Purpose, technique, decision, action and result

When drawing up a lesson plan for a mathematical topic, teachers may pursue a general or specific purpose in their treatment of student error. General purposes may often be narrowed down into other more specific purposes. In such cases, their decision consists in such purposes. For instance, to reach a general purpose such as

- P: to help students overcome errors committed in a given topic, teachers may decide to pursue a more specific purpose such as
- P1: to sequence mathematics tasks to address errors gradually.

Purposes may be successively narrowed, yielding a series of tiered, increasingly specific purposes until the only options available are specific actions implemented by the teacher to attain a result. For instance, when teachers pursue the specific purpose of sequencing tasks (P1), the decision involves choosing between the following actions:

- A1: introducing tasks in increasing order of difficulty, with the ones that induce students to err appearing at the end of the sequence only, or

Received: 24-04-2014 Accepted: 22-05-2015

- A2: distributing tasks in a way that confronts students with errors throughout the lesson.

Implementing an action yields a result. Here, the result of action A1 is a list of tasks with the simplest at the beginning and the most complex at the end, while the outcome of action A2 is a list containing complex tasks from start to finish. While a specific purpose and an action may be couched in similar terms, they differ in that the latter generates a result.

A technique is a suite of routine procedures supported by sound reasoning and intended to solve a problem. Two types of techniques are depicted in Figure I: (a) decision-making techniques (type 1) and (b) action implementation techniques (type 2). One decision-making technique is to review the alternatives available and select one of the options. Action implementation techniques enable teachers to generate a result from the option chosen. To implement tasks to help students overcome errors, for instance, teachers may first review the various options available and choose one: e.g., to design teaching aids for students as they perform a given task. In that review, a decision-making technique, they compare the effect of such aids and of other options on students' ability to overcome errors. After adopting a decision and generating an action (designing aid), teachers deploy techniques to perform the action and obtain a result. They may, for instance, analyse every mistake students are expected to make and prepare questions that alert them to their error. The result of such an action, implemented with such a technique, would be a list of error-associated questions.

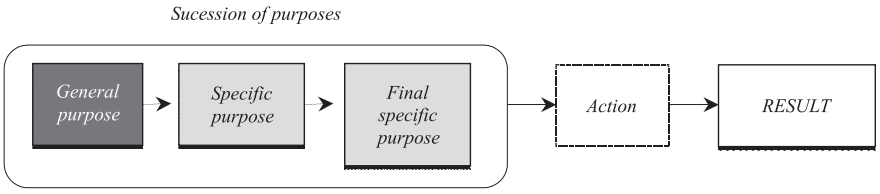
This conceptual framework underlies the definition of what is meant here by use of error, discussed below, along with the focus of the present study.

Use of error and focus of study

Further to the above conceptual framework, use of error is regarded here as a succession of teacher purposes involving the notion of error, which lead to an action and the result of that action (Figure II). Actions, which determine how errors are used, are associated with the last of a succession of purposes to be attained by a teacher. They generate results that have implications for lesson planning or implementation. As Figure

I shows, when addressing a general purpose, teachers may decide to implement an action or formulate one or a series of more specific purposes. Consequently, the succession of purposes begins with a general purpose and may entail the definition of one or more narrower specific purposes. The use of error is characterised by three components: the succession of purposes, the action associated with the final specific purpose and the result of that action.

FIGURE II. Sequence in use of error



This empirical study identified the uses of error applied by a group of participants in a training programme for practising mathematics teachers.

Methodology

This section describes the context of the empirical study, the participants, the sources of information used and the data coding process.

The training programme for mathematics teachers in which this research was conducted covered mathematics lesson planning, implementation and evaluation. Over a two-year period, participants worked in groups of four that systematically analysed a mathematical topic from different standpoints (including systems of representation, common errors and associated phenomenology). They determined the learning objectives pursued, designed a task sequence to attain their objectives, predicted students' performance in the tasks assigned, designed observation tools to evaluate learning and teaching, implemented the curriculum in place in their educational institutions,

Received: 24-04-2014 Accepted: 22-05-2015

assessed the relevance and effectiveness of the plan implemented, and established a new and improved design. Each group wrote up a report on these matters and formulated a lesson plan for a mathematical topic. The 100-page reports included the analysis of the mathematical topic, the decisions adopted by the groups during lesson formulation and their justification[2]. This description of decisions and justifications provided insight into the purposes pursued by the teachers, the actions proposed and the results obtained. As learning errors may appear at any stage of the process, these reports were used as sources of information to identify the use of error.

The study was conducted with the first edition of the programme, in which 26 secondary school mathematics teachers, divided into six groups, participated. The topics addressed by the groups included addition and subtraction of whole numbers, linear equations (two groups), a graphical method for solving 2x2 linear equations, and trigonometric ratios (two groups). The teachers granted their written consent to participate in the study.

Basic categories were established to analyse the groups' reports, based on the above conceptual framework for decision-making: the purpose defined by the group of teachers, the action proposed or the result obtained. All the sections in the reports containing references to errors were extracted and the elements matching the analytical categories were identified in each extract.

The following is an example of one of the extracts selected.

> *When students are confronted with an error or difficulty, the teacher should guide them with questions that elicit reasoning around the action taken, to overcome the error.*

Here the purpose identified was to "overcome the error" and the action "guide with questions". The results, which were not explicitly stated, would have entailed listing the specific questions to be posed.

Each extract was coded into a triad in terms of the above three factors (purpose, action, result), in which one or two of the fields could be void of content. These triads were then organised and inter-related by means of a cyclic procedure in which triads with similar purposes, actions or

(2) Group reports can be downloaded on http://is.gd/PAUfQ5.

Received: 24-04-2014 Accepted: 22-05-2015

results were grouped together. The resulting groups were then ranked by purpose, distinguishing general from specific purposes and the sets of actions associated with each specific purpose were identified. The results associated with a given action were likewise grouped. This procedure yielded more refined analytical categories. Based on these new categories, the raw data were reviewed again to re-code the extracts and verify the validity of the new categories and their inter-relationships. This cyclical process of using the raw data to refine basic categories, in turn validated by re-coding the data, led to successive category refinement and structuring. The process was completed when the groups of purposes, actions and results generated no longer shared sufficient elements to be re-grouped.

Lastly, the uses of error were established as discussed in the following section, in which the extracts were matched to the refined categories obtained. Table 1 contains four related texts which together exemplify the use of error. In extract A, the teacher trainees defined a general purpose consisting in overcoming errors. In extract B, they narrowed this purpose to the specification of tasks that would induce students to err. In extract C they reviewed the tasks. Lastly, in extract D they reported that a specific task, which they called the Chicago Wheel, was chosen to overcome a specific error. Although the four extracts were found in separate sections of the report, they were associated with triads that were grouped together and ranked. The outcome was the description of one use of error: overcoming errors, in which the teachers proposed to define suitable tasks, implemented the respective action (i.e., choice of a task) and attained a result, namely the specification of the task chosen.

Received: 24-04-2014 Accepted: 22-05-2015

TABLE 1. Coding extracts that together constitute use of error

Id.	Extract	Purpose	Action	Result
A	"When planning […] we sought to enhance learning expectations for the lesson with the inclusion of overcoming difficulties."	To overcome errors		
B	"Errors were one of the key guides in task formulation."	To define tasks that would induce students to err		
C	"We specify tasks using the information obtained from our recently introduced instruction analysis organisers."		The group chose tasks	
D	"In the Chicago Wheel task, when asking for the answer to the initial situation […] the students are asked […] to apply the results to the problem posed. The idea is to overcome error E7.4."	To overcome errors		Specific task

Ranking the categories and hence the extracts was not an automatic exercise. Rather, it called for intense interpretation and justification by the researchers who, on the one hand, drew from their understanding of the training programme at issue and on the other reviewed the extracts in the context of the conceptual framework defined for the analysis.

Results: description of the uses of error

This section describes how the purpose, action and result categories were ranked. To facilitate the presentation of the findings, the ranks were matched to the description of the 16 uses of error identified.

The basic purpose category yielded the following three general purposes for which the teachers in the sample used error in lesson plans for mathematical topics:

- to enable students to overcome topic-related errors,
- to evaluate students' cognitive knowledge, and
- to generate information useful for other planning-related questions.

Received: 24-04-2014 Accepted: 22-05-2015

Each of these three purposes delimits a set of uses of error that were specified in different ways and at different times in the planning process. These uses are described in the following three sub-sections in conjunction with the purposes, actions or results appearing in the reports analysed.

Uses associated with overcoming topic-related errors

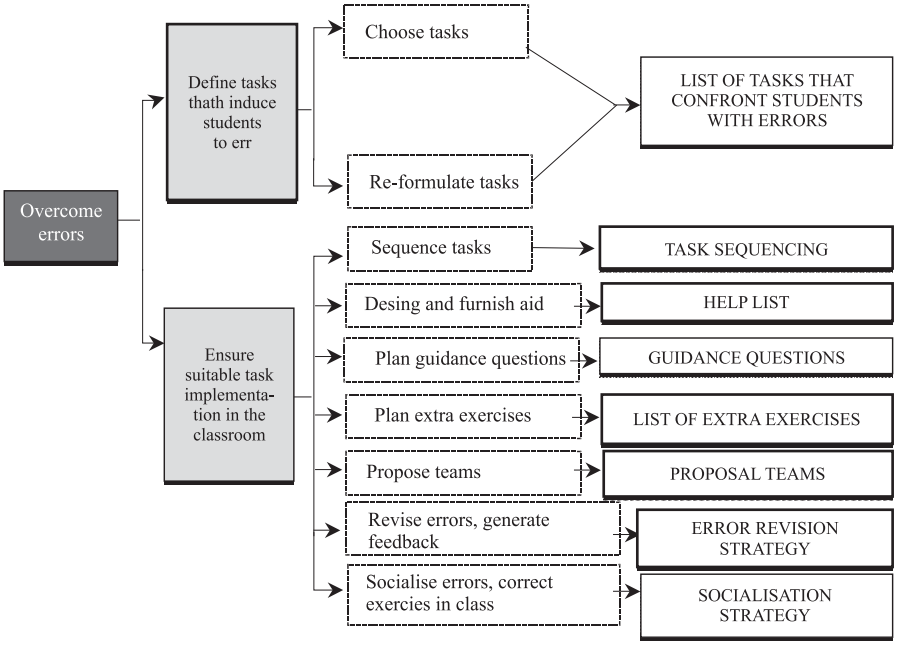
Overcoming the errors associated with a topic was the general purpose defined most frequently by teachers participating in the training programme described here. They expressed this purpose in their final report in extracts such as the one reproduced below.

> *The lesson discussed below addresses the elements that we deem necessary to solve the problems encountered by seventh-year students in situations involving adding and subtracting whole numbers.*

The flowchart in Figure III below shows the action categories and respective results. Eight uses of error were established under this general purpose. Some of them are exemplified in the discussion that follows with extracts from the teachers' final reports.

Received: 24-04-2014 Accepted: 22-05-2015

FIGURE III. Uses of error for purpose 1



The general purpose, overcoming errors, was re-formulated as two specific purposes. The first was to define tasks that induce students to err. This gave rise to two types of actions: choosing the tasks from a set of given tasks and re-formulating existing tasks. In the following extract, for instance, teachers formulated part of a task with the explicit intention of overcoming an error.

> *To overcome error E4, with which we associated no skills, in the second phase [of the Chicago Wheel] we introduced work with identities, so that students could infer and generalise the results.*

The two actions (select and re-formulate tasks) led to the same result: a list of tasks that confronted students with error.

The second specific purpose was to plan task implementation in the classroom in ways that would help overcome error. The extract below

Received: 24-04-2014 Accepted: 22-05-2015

shows how the teachers, while pursuing the general purpose, i.e., to help students overcome errors, focused on the more specific purpose of ensuring satisfactory classroom delivery.

> *The teacher's delivery should be dynamic, posing questions around team actions and discussion so that she controls when and to what extent she leaves the student teams on their own to handle the difficulties arising during the exercise and its possible solution... When students are confronted with an error or difficulty, the teacher should guide them with questions that prompt reasoning around the action performed, to overcome the error.*

This purpose prompted teachers to take several actions. The first was to sequence tasks, whose outcome was a specific order in which they were assigned. One group of teachers, for instance, proposed to begin with simple tasks involving no error and, once the basic knowledge was acquired, to introduce more complex tasks that induced error. The other actions found in the raw data included designing and offering assistance, preparing guidance questions, preparing extra exercises, proposing student teamwork, revising errors and generating feedback, socialising errors and correcting exercises in the classroom. These actions translated into the preparation of specific actions to be implemented during the performance of classroom tasks. The following extract, for example, illustrates the decision adopted by a group of teachers to design and offer assistance to improve task performance and hence overcome errors.

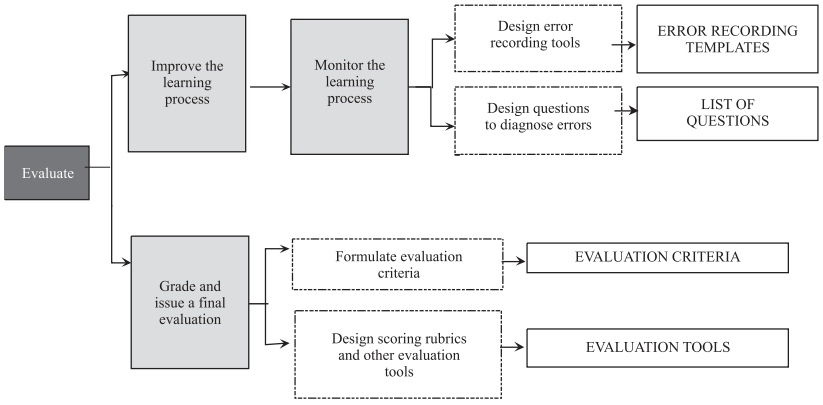
> *We included aids that could optimise task performance and address the errors students might commit.*

Uses associated with evaluating students' cognitive knowledge

The errors committed were used to evaluate students' cognitive knowledge of the mathematical topic. This purpose was broken down into two other purposes which in turn yielded a series of actions that involved the four uses of error graphed in Figure IV and explained below.

Received: 24-04-2014 Accepted: 22-05-2015

FIGURE IV. Uses of error for purpose 2



Evaluation was broken down into two further purposes. The first was to improve learning by assessing training. This led to monitoring the learning process, which entailed two types of actions. The first was to design tools for recording the errors arising during the exercise. In the two extracts reproduced below teachers described the two tools designed: an observation grid and a check-list.

> *The observation grid is a classroom format that contains the planned learning pathway[3], possible student errors and action that can be taken.*
> *The check-list is used to monitor each skill included in the learning pathway and record the errors observed […].*

The second action was to design specific questions to determine whether students had overcome certain errors at a given stage of the exercise. The second specific purpose was to issue a final evaluation of students' cognitive knowledge after a given training period. In the

(3) The term learning pathway cited in the following extracts is defined in the training programme as student strategies to perform mathematical tasks.

Received: 24-04-2014 Accepted: 22-05-2015

following extract, for instance, teachers used the appearance of errors as proof that students had not mastered a given skill.

> *We thought that with this task we would contribute to symbolic language and representation skills. In connection with the former, we concluded... [that] there was no evidence that the class had corrected the errors committed in algebraic expression.*

To fulfil this specific purpose, errors were used to establish evaluation criteria for a topic and design the respective evaluation tools. The following extract is an example of an evaluation criterion partially based on the observation of errors.

> *a high performer exhibited suitable skills in this activity and followed the learning pathway with apparent ease, particularly in connection with algorithms, but committed errors due to the application of irrelevant rules or strategies.*

Uses associated with generating information useful for other planning-related questions

Errors committed in mathematical topics were used to generate information supporting other planning-related questions. The data analysed showed that this general purpose was broken down into two narrower purposes: formulating learning expectations and formulating learning hypotheses. The teachers decided to undertake a number of actions that involved the four uses of error represented in Figure V and explained below.

Learning expectations were formulated under actions in which new skills were acquired by building on errors, certain special skills associated with serious errors were specifically identified and objectives were re-formulated to embrace the need to overcome certain errors. The following extract shows that teachers used errors to formulate specific skills described in their reports.

> *Errors were one of the key guides in task formulation. Some errors prompted the formulation of skills that were subsequently introduced*

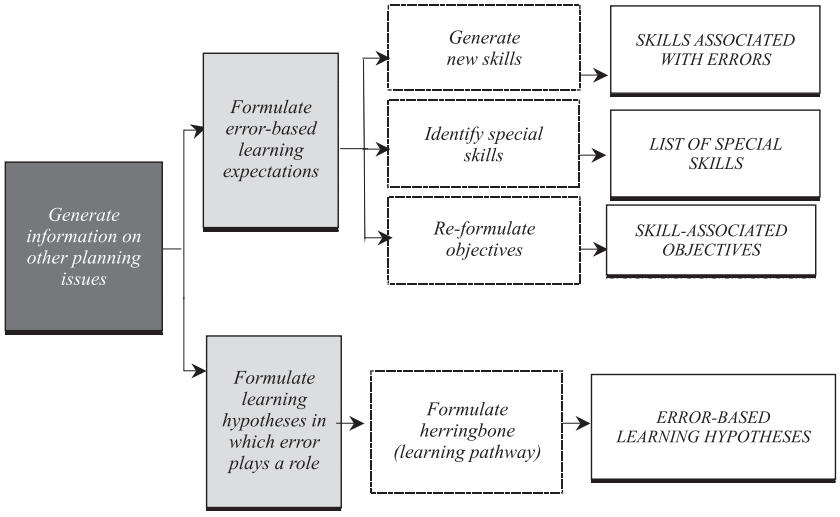
Received: 24-04-2014 Accepted: 22-05-2015

in tasks. Error E7.4, for instance, induced the formulation of skill C12…

In the formulation of learning hypotheses in which error plays a role, the action taken was to build errors into what was denominated the "herringbone" learning pathway. That format explicitly included the stages in task performance at which students were expected to commit errors. The following extract shows how error was incorporated into learning pathways.

When planning the lesson we identified certain learning errors in the topic dealing with trigonometric ratios. These errors were included in the learning pathways, with a prediction of the stages in which students were expected to be unsuccessful.

FIGURE V. Uses of error for purpose 3



Received: 24-04-2014 Accepted: 22-05-2015

Conclusions

This article proposes a conceptual framework for decision-making to explore the use of error in mathematics teacher training. Taking that framework as a basis, uses of error were identified in the final reports drafted by groups of teachers participating in a training programme. The uses of error were determined by relating triads that shared a given purpose, contained actions associated with a shared purpose and specified the results of such actions. The triads were then ranked on those grounds.

The analysis of the groups' output revealed a broad spectrum of uses (16 in all), which were classified under three general purposes: to overcome errors, to evaluate students' cognitive knowledge and to generate information useful for other planning-related questions. A substantial number of actions and results arose in connection with error at different stages of planning. These actions involved all areas of teaching: learning objectives, student tasks, classroom delivery, attention to diversity, student evaluations and training. This variety is remarkable, given that the training programme did not focus in particular on the use of learning errors in lesson planning. Error was just one of a series of educational conceits addressed in the programme.

The present study supplements earlier findings and pursues the respective research in greater depth. The groups of teachers participating in the present training programme, unlike the teachers studied by Santagata (2005) or the U.S. staff in Schleppenbach et al.'s (2007) survey, did not set out to directly correct or prevent error. The groups studied here focused on the choice and implementation of tasks that would induce students to commit learning errors routinely observed in connection with a given mathematical topic. Their approach to error was to generate cognitive conflicts that would help students modify their partial knowledge. The uses of error observed in the present study to be related to the general purpose of overcoming errors supplement the proposals for educational actions identified by Son and Sinclair (2010) and Son (2013). In addition, the present study identified and characterised two further general purposes related to the use of error, as discussed above.

The present findings are relevant to the design and implementation of teacher training programmes. The diversity of the use of error in

Received: 24-04-2014 Accepted: 22-05-2015

teacher training observed supports the premise that this practice should be explicitly addressed in the development of teachers' professional knowledge. The proposed structure, with general and specific purposes and their respective actions and results may serve as a guide for the design, development and monitoring of teacher training activities.

Limitations and future studies

The present study explored the uses made of error by six groups of teachers training in a specific context. Therefore, the findings cannot be generalised: other groups of teacher trainees might or might not use error in the same ways and the uses identified are not necessarily the only ones possible.

The information available limited the analysis to the participants' written reports. The nature of that information ruled out research into two issues that the authors aim to explore in future.

The first is error use monitoring during the classroom delivery of the lessons formulated. This issue is regarded as especially relevant because teachers prepare error-associated tasks and develop tools to monitor their implementation. The way that such tools are used in a new context in which teachers improvise decisions sheds light on the use of error actually present in the classroom and the relationships among teachers' decision-making processes. Such research calls, on the one hand, for interviews with teachers and analysis of classroom videos, and on the other, for re-formulating the conceptual framework for adaptation to spontaneous decisions (not foreseen in lesson plans) that arise in the classroom.

The second question deals with the techniques associated with the use of error. While the conceptual framework proposed covers technique, the detailed identification of techniques in decision-making calls for analysing teachers' interactions when adopting decisions. Such analysis entails classroom audio and video recordings of their interactions. This type of research is relevant to characterising the techniques associated with the use of error, i.e., identifying rationally designed procedures that induce teachers to select a given option and implement the actions arising from the option selected.

Received: 24-04-2014 Accepted: 22-05-2015

References

An, S. y Wu, Z. (2012). Enhancing mathematics teachers' knowledge of students' thinking from assessing and analyzing misconceptions in homework. *International Journal of Science and Mathematics Education, 10*(3), 717-753.

Bishop, A. J. (1976). Decision-making, the intervening variable. *Educational Studies in Mathematics, 7*(1), 41-47.

Borasi, R. (1994). Capitalizing on errors as «springboards for inquiry»: A teaching experiment. *Journal for Research in Mathematics Education*, *25*(2), 166-208.

Borasi, R. (1996). *Reconceiving mathematics instruction: A focus on errors*. Norwood, NJ: Ablex Publishing Corporation.

Borko, H., Roberts, S. A. y Shavelson, R. (2008). Teachers' decision making: From Alan J. Bishop to today. En P. Clarkson y N. Presmeg (Eds.), *Critical issues in mathematics education* (pp. 37-67). New York, NY: Springer.

Brodie, K. (2014). Learning about learner errors in professional learning communities. *Educational Studies in Mathematics, 85*(2), 221-239.

Brousseau, G. (2001). Les erreurs des élèves en mathématiques : Etude dans le cadre de la théorie des situations didactiques. *Petit x, 57*, 5-30.

Call, J. J. (2012). *Student Teachers' Interactive Decisions with Respect to Student Mathematics Thinking*. Tesis de doctorado. Brigham Young University, Provo, UH.

Corbin, J. y Strauss, A. (1990). Grounded theory research: procedures, canons, and evaluative criteria. *Qualitative Sociology*, *13*(1), 3-21.

Evans, J. St. B. T., Over D. E., y Handley S. J. (2003). A theory of hypothetical thinking. En D. Hardman y L. Macchi (Eds.): *Thinking: psychological perspectives on reasoning, judgment and decision making* (pp. 3-21). Chichester, UK: John Wiley.

Garuti, R., Boero, P. y Chiappini, G. (1999). Bringing the voice of Plato in the classroom to detect and overcome conceptual mistakes. En O. Zaslavsky (Ed.), *Proceedings of the 23 th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 3-9). Haifa, Israel: PME.

Hansson, S. O. (2005). *Decision theory. A brief introduction*. Documento no publicado. Stockholm, Suecia: Royal Institute of Technology (KTH).

Received: 24-04-2014 Accepted: 22-05-2015

Heinze, A. y Reiss, K. (2007). Mistake-handling Activities in the Mathematics Classroom: Effects of an In-service Teacher Training on Students' Performance in Geometry. En J. H. Woo, H. C. Lew, K. S. Park y D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 31, pp. 9-16). Seoul, Corea: PME.

Kahneman, D., Slovic, P. y Tversky, A. (1982). *Judgment under uncertainty: Heuristics and biases*. Cambridge, MA: Cambridge University Press.

Keith, N. y Frese, M. (2008). Effectiveness of error management training: A meta-analysis. *Journal of Applied Psychology, 93*(1), 59.

Lannin, J. K., Barker, D. D. y Townsend, B. E. (2007). How students view the general nature of their errors. *Educational Studies in Mathematics, 66*(1), 43-59.

NCTM. (2000). *Principles and standards for school mathematics*. Reston: NCTM.

Prediger, S. (2010). How to develop mathematics-for-teaching and for understanding: the case of meanings of the equal sign. *Journal of Mathematics Teacher Education, 13*(1), 73-93.

Rach, S., Ufer, S. y Heinze, A. (2013). Learning from errors: effects of teachers training on students' attitudes towards and their individual use of errors. *PNA, 8*(1), 21-30.

Rich, P. y Hannafin, M. (2008). Decisions and reasons: examining preservice teacher decision-making through video self-analysis. *Journal of Computing in Higher Education*, *20*(1), 62-94.

Rico, L. (1997). Reivindicacio´n del error en el aprendizaje de las matema´ticas. *E´psilon, 38*, 185-198.

Santagata, R. (2005). Practices and beliefs in mistake-handling activities: A video study of Italian and US mathematics lessons. *Teaching and Teacher Education, 21*(5), 491-508.

Savage, L. J. (1951). The theory of statistical decision. *Journal of the American Statistical association, 46*(253), 55-67.

Schifter, C. C., Natarajan, U., Ketelhut, D. J., & Kirchgessner, A. (2014). Data-Driven Decision Making: Facilitating Teacher Use of Student Data to Inform Classroom Instruction. *Contemporary Issues in Technology and Teacher Education*, 14(4). Disponible en http://www.citejournal.org/vol14/iss4/science/article1.cfm

Schleppenbach, M., Flevares, L. M., Sims, L. M. y Perry, M. (2007). Teachers' responses to student mistakes in Chinese and US mathematics classrooms. *The elementary school journal, 108*(2), 131-147.

Schoenfeld, A. (2011). Toward professional development for teachers grounded in a theory of decision making. *ZDM, 43*(4), 457-469.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. New York, NY: Routledge.

Shavelson, R. J., Webb, N. M. y Burstein, L. (1986). Measurement of teaching. En Wittrock (Ed.), *Handbook of research on teaching* (pp. 50-91). New York: Macmillan.

Simon, M. y Schifter, D. (1991). Towards a constructivist perspective: An intervention study of mathematics teacher development. *Educational Studies in Mathematics, 22*(4), 309-331.

Son, J.-W. (2013). How preservice teachers interpret and respond to student errors: ratio and proportion in similar rectangles. *Educational Studies in Mathematics, 84*(1), 49-70.

Son, J.-W. y Crespo, S. (2009). Prospective teachers' reasoning and response to a student's non-traditional strategy when dividing fractions. *Journal of Mathematics Teacher Education, 12*(4), 235-261.

Son, J.-W. y Sinclair, N. (2010). How Preservice Teachers Interpret and Respond to Student Geometric Errors. *School Science and Mathematics, 110*(1), 31-46.

Stefaniak, J. y Tracey, M. (2014). An examination of the decision-making process used by designers in multiple disciplines. *TechTrends: Linking Research & Practice to Improve Learning*, *58*(5), 80-99.

Contact address: María José González. Universidad de Cantabria, Facultad de Ciencias. Departamento de Matemáticas, Estadística y Computación. Facultad de Ciencias, Universidad de Cantabria. Av. Los Castros s/n. 39005 Santander, España. E-mail: gonzalelm@unican.es

Received: 24-04-2014 Accepted: 22-05-2015

La *Revista de Educación* es una publicación científica del Ministerio de Educación, Cultura y Deporte español. Fundada en 1940, y manteniendo el título de *Revista de Educación* desde 1952, es un testigo privilegiado de la evolución de la educación en las últimas décadas, así como un reconocido medio de difusión de los avances en la investigación y la innovación en este campo, tanto desde una perspectiva nacional como internacional. La revista es editada por la Subdirección General de Documentación y Publicaciones, y actualmente está adscrita al Instituto Nacional de Evaluación Educativa de la Dirección General de Evaluación y Cooperación Territorial.

GOBIERNO DE ESPAÑA
MINISTERIO DE EDUCACIÓN, CULTURA Y DEPORTE

NIPO línea: 030-15-016-X
NIPO ibd: 030-15-017-5
ISSN línea: 1988-592X 0034-8082
ISSN papel: 0034-8082

www.mecd.gob.es/revista-de-educacion